Laurea in Chimica e Tecnologie Farmaceutiche

corso di **Matematica e statistica** a.a. 2016/17

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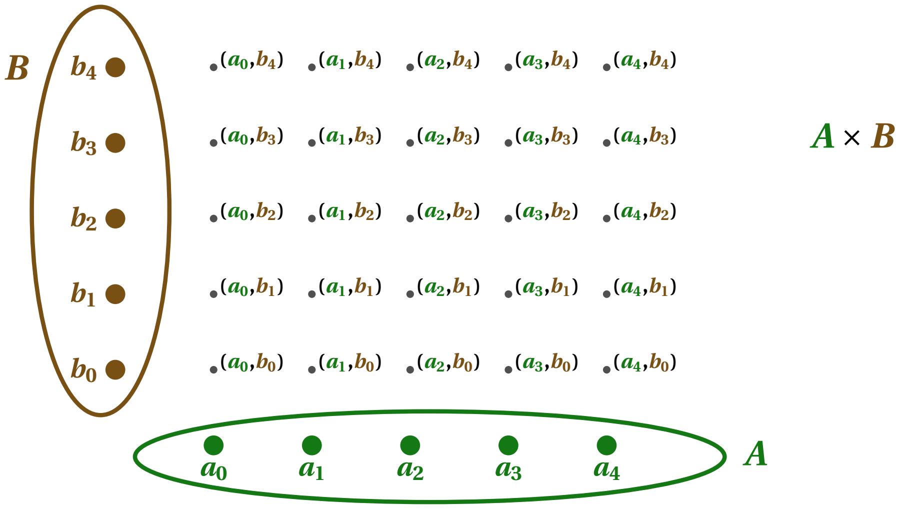
Sito Moodle: http://elearning.moodle2.unito.it/dstf/course/view.php?id=295

1 Cartesian products and relations

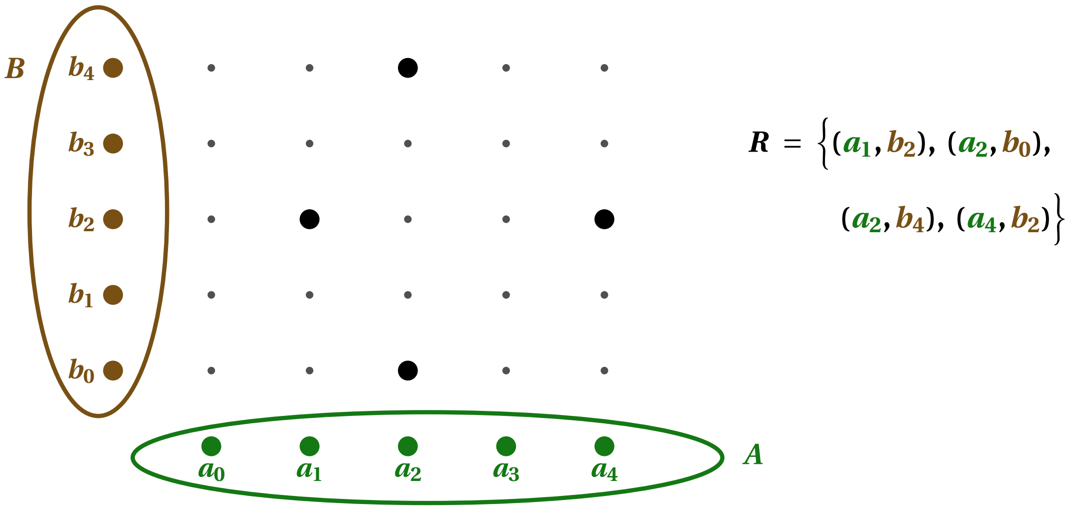
Let A and B be two sets. We denote by $\boldsymbol{A \times B}$ the sets of ordered pairs such that the first elements belongs to A and the seconds belongs to B. We write $\boldsymbol{A^2}$ for $A \times A$. We call $A \times B$ the **Cartesian product** of A and B; we call A^2 a **Cartesian power** of A.



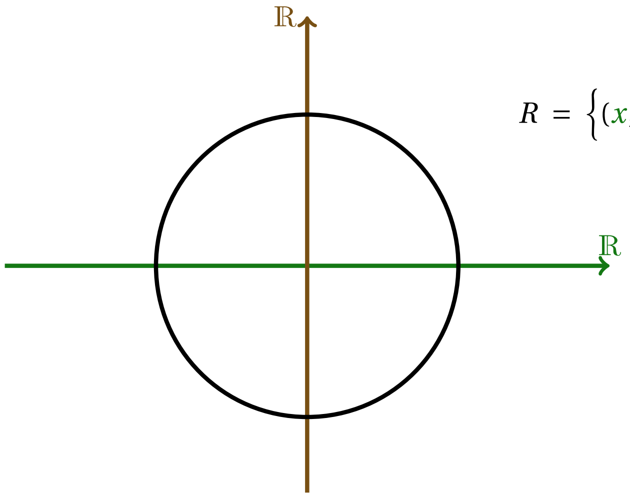
Let A_i, for $i = 1, \ldots, n$, be sets. We denote by $\boldsymbol{A_1 \times \cdots \times A_n}$ the set of n-tuples such that the i-th elements belongs to A_i. If $A_i = A$ for all i, we write A^n. It is mostly harmless (and sometimes also convenient) to identify the pair $\big((a_1, \ldots, a_n), a_{n+1}\big)$ with the $n+1$-tuple $(a_1, \ldots, a_{n+1})$. Consequently we may identify $\big(A_1 \times \cdots \times A_n\big) \times A_{n+1}$ with $A_1 \times \cdots \times A_{n+1}$.

A subset of $A \times B$ is called a **binary relation between** A and B. When $B = A$ we say: binary relation **on** A. An ***n*-ary relation** is a subset of $A_1 \times \cdots \times A_n$. By the observation above, every $n+1$-ary relation is associated to a binary relation between $A_1 \times \cdots \times A_n$ and A_{n+1}.

We can represent a relation using the picture of the Cartesian product as given above.

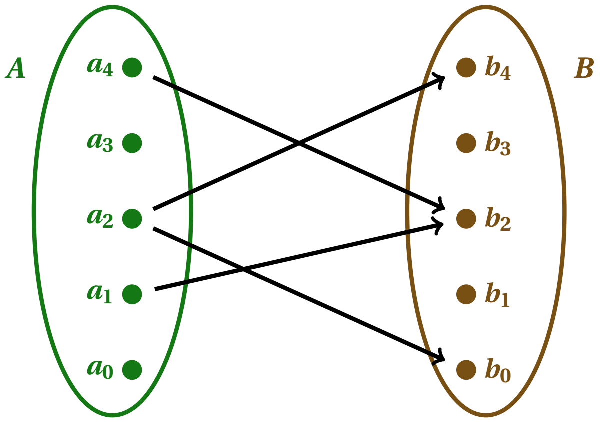


We are acquainted with this representation when R is a relations on $\mathbb{R}$. For eample,



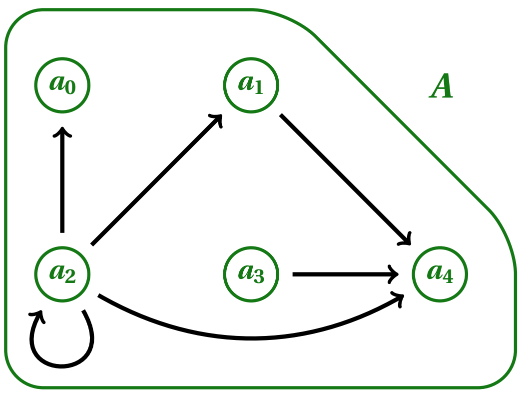
$$R \;=\; \left\{(x, y) \;:\; x^2 + y^2 = 1\right\}$$

Another way to represent a binary relation may be used when A and B are finite. We can draw an arrow for each pair $(a, b) \in R$



$$\boldsymbol{R} \;=\; \left\{(\boldsymbol{a_1}, \boldsymbol{b_2}),\ (\boldsymbol{a_2}, \boldsymbol{b_0}),\ (\boldsymbol{a_2}, \boldsymbol{b_4}),\ (\boldsymbol{a_4}, \boldsymbol{b_2})\right\}$$

When $A = B$, i.e. when R is a relation on a finite set A, then R can be represented by a diagram as the one below.



$$\boldsymbol{R} \;=\; \left\{(\boldsymbol{a_1}, \boldsymbol{a_4}),\ (\boldsymbol{a_2}, \boldsymbol{a_0}),\ (\boldsymbol{a_2}, \boldsymbol{a_1}),\ (\boldsymbol{a_2}, \boldsymbol{a_4}),\ (\boldsymbol{a_3}, \boldsymbol{a_4}),\ (\boldsymbol{a_2}, \boldsymbol{a_2})\right\}$$

If $R \subseteq A \times B$ is a binary relation, we define

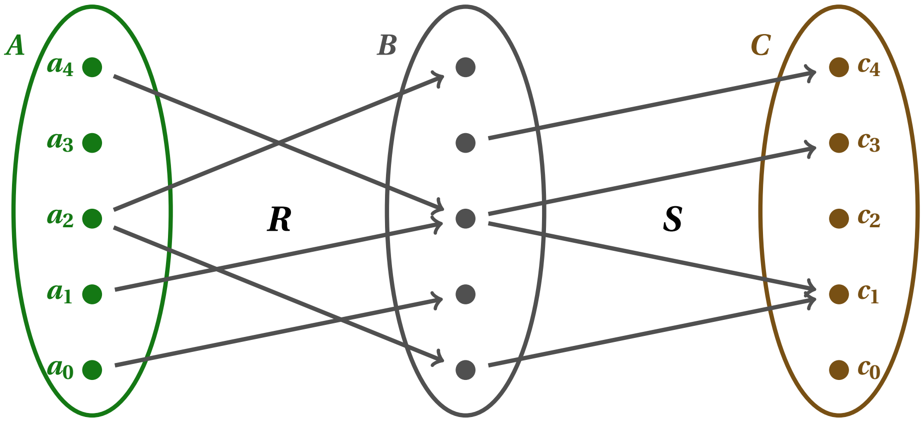
$$\boldsymbol{R^{-1}} \;=\; \{(y, x) \;:\; (x, y) \in R)\} \;=\; B \times A$$

This is called the **inverse of *R***. A representation of R^{-1} is obtained by reversing the arrows in the diagrams above.

Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be two bynary relations. The **composition** of R and S is the relation

$$\boldsymbol{S \circ R} \quad = \quad \Big\{(x,y) \,:\, \exists z \,\big[(x,z) \in R \wedge (z,y) \in S)\big]\Big\}.$$

With reference to the picture below, $S \circ R$ contains a pair (a, c) if there is a path from a to c. Note that the order in which we read $S \circ R$ is from right to left.



$$\boldsymbol{S \circ R} \quad = \quad \{(a_1, c_1),\ (a_1, c_3),\ (a_2, c_1),\ (a_4, c_1),\ (a_4, c_3)\}$$

2 Functions

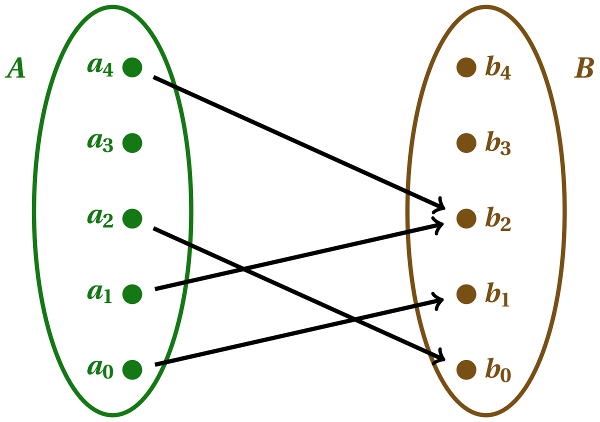
A **(unary) function** is binary relation R such that $\forall x\, \exists^{=1} y\, (x,y) \in R$. Hence, the relations R and S above are not functions. The letters $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$, are typically used for functions.

If f is a function, the unique y such that $(x,y) \in f$ is denoted by $\boldsymbol{f(x)}$. But it is very common to denote by $\boldsymbol{f(x)}$ the whole function f. This because the symbol x is often intended as a place-holder for some input not as the input itself.

We write $\boldsymbol{f : A \to B}$ to say that $f \subseteq A \times B$ and f is a function. The triple (f, A, B) is called a **map**. We call A the **domain** and B the **codomain** of the map.

The set $\mathbf{dom}\boldsymbol{f} = \{x \,:\, \exists y\, (x,y) \in f\}$ is called the **domain of definition** of the function f. The set $\mathbf{range}\boldsymbol{f} = \{y \,:\, \exists x\, (x,y) \in f\}$ is called the **range** (or the **image**) of f. Hence the domain of definition of f is always a subset of the domain of the map and the range is always a subset of the codomain.

If the domain coincides with the domain of definition we say that the map is **total**, otherwise that it is **partial**. If the range coincides with the codomain we say that the map is **surjective**.



$$\text{dom} f = \{a_0, a_1, a_2, a_4\}$$
$$\text{range} f = \{b_0, b_1, b_2\}$$

If $f : A \to B$ and $C \subseteq A$, we write $f_{\restriction C}$ for the function $f \cap (C \times B)$. In words we say that $f_{\restriction C}$ is the **restriction of f to C**. Note that any $g \subseteq f$ is a restriction of f, in fact $g = f_{\restriction \text{dom} g}$.

If $A = A_1 \times \cdots \times A_n$ then we say that f is an **n-ary function**. We write $f(x_1, \ldots, x_n)$ for the unique y such that $\big((x_1, \ldots, x_n), y\big) \in f$. When convenient we can identify an n-ary function with a subset of $A_1 \times \cdots \times A_n \times B$, that is an $n+1$ anty function.

Again note that $f(x_1, \ldots, x_n)$ can also be used to mean the whole function. This notation helps to remind arity of function.

The relation f^{-1} may not be a function. When f^{-1} is a function we say that f is **injective** (or **one-to-one**). It is easy to see that f is injective if and only if

$$\forall x, y \in \text{dom} f \ \big[x \neq y \to f(x) \neq f(y)\big].$$

Note that when f is injective $f^{-1} \circ f = \text{id}_{\text{dom} f}$ and $f \circ f^{-1} = \text{id}_{\text{range} f}$.

Warning. The terminology introduced above is not adopted uniformly. Every area of mathematics has a slightly different parlance. For instance in real analysis the domain is always $\mathbb{R}$ (or $\mathbb{R}^n$) and the codomain is always $\mathbb{R}$. So it is common to denote a function simply by $f(x)$ (or $f(x_1, \ldots, x_n)$). When it is necessary to refer to the *relation* associated to the map, one says the **graph of $f(x)$**.

Let $f : A \to B$ be a map and let $C \subseteq A$. The **image of C under f** is the set

$$f[C] = \{f(x) \ : \ x \in C\}.$$

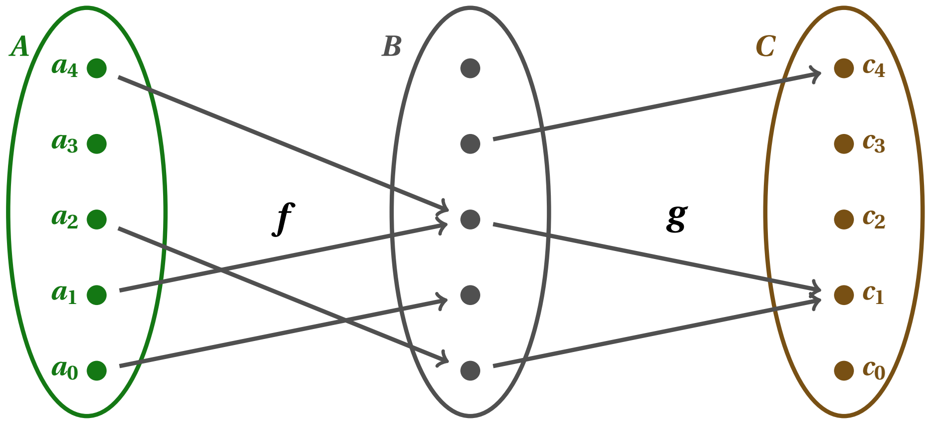
If $D \subseteq C$, the **inverse image of C under f** is the set

$$f^{-1}[D] = \big\{x \ : \ f(x) \in D\big\}.$$

Note that $f^{-1}[D]$ is well-defined also when f is not injective. Note that $f[A] = \text{range} f$ and $f^{-1}[B] = \text{dom} f$.

3 Composition of maps

Let $f: A \to B$ and $g: B \to C$ be to maps and assume that the codomain of the frst map is the domain of the second. Their **composition** is the map $g \circ f: A \to B$. When domain and codomain are clear from the context we may write $g\big(f(x)\big)$ for the composition.



$$\boldsymbol{g \circ f} \quad = \quad \{(\boldsymbol{a_1}, \boldsymbol{c_1}),\ (\boldsymbol{a_2}, \boldsymbol{c_1}),\ (\boldsymbol{a_4}, \boldsymbol{c_1})\}$$

Note that $\operatorname{dom}(g \circ f) = f^{-1}[\operatorname{dom} g] = \{a_1, a_2, a_4\}$

4 A few important partial maps $f: \mathbb{R} \to \mathbb{R}$

$$
\begin{array}{lll}
f(x) = \dfrac{1}{x} & \operatorname{dom} f = \mathbb{R} \smallsetminus \{0\} & \operatorname{range} f = \mathbb{R} \smallsetminus \{0\} \\
f(x) = \sqrt{x} & \operatorname{dom} f = [0, \infty) & \operatorname{range} f = [0, \infty) \\
f(x) = \ln x & \operatorname{dom} f = (0, \infty) & \operatorname{range} f = \mathbb{R} \\
f(x) = \arcsin(x),\ \arccos(x) & \operatorname{dom} f = [-1,\ 1] & \operatorname{range} f = [-\pi,\ \pi] \\
f(x) = \tan(x) & \operatorname{dom} f = \mathbb{R} \smallsetminus \left\{\dfrac{\pi}{2} + k\pi\ :\ k \in \mathbb{Z}\right\} & \operatorname{range} f = \mathbb{R} \\
f(x) = \cot(x) & \operatorname{dom} f = \mathbb{R} \smallsetminus \left\{k\pi\ :\ k \in \mathbb{Z}\right\} & \operatorname{range} f = \mathbb{R}
\end{array}
$$