Exercises on Ordinary Differential Equations

0.1 Direct methods

Write the solutions to the following ODE problems for the function $y(x)$ using the methods discussed in the lectures. The methods are: separation of variables, the potential method, and methods for homogeneous and inhomogeneous linear equations, in particular variation of constants.

Problems

1) Find the general solution to $xy'(x) = 1 + y(x)$;
2) Find the solution to $xy' - xy = y$ such that $y = 1$ for $x = 1$;
3) Write the general solution to $y'\left(y + x^2\right) + 2xy + \sin(x) = 0$.
4) Consider the ODE $y'' + 2y' + y = e^{-x}$. A particular solution is given by $y = \frac{x^2 e^{-x}}{2}$. Write the general solution.
5) Find the general solution to $y' - xy = 1$.
6) Solve $(1 + y^2) + xyy' = 0$ with $y = 0$ when $x = 5$.
7) Write in implicit form the general solution to $y'(xe^y + 1) + x^2 + e^y = 0$.
8) Find the solution to $y' + x^2 y = x^2$ such that $y = 3$ for $x = 0$.
9) Consider the ODE: $y'' + 2xy' - 2y = 0$. A solution is given by $y(x) = x$. Write the general solution.
10) Solve $(\cos(x) + 1)y' - (y + 1)\sin(x) - 2x = 0$, with $y(0) = 0$.

Solutions and hints:

1) Hint: separable equation. Solution: $y(x) = Ax - 1$.
2) Hint: separable. Solution: $y(x) = e^{-1+x}x$.
3) Hint: use the potential method. Solution: $y(x) = \pm\sqrt{A + x^4 + 2\cos(x) - 1} - x^2$.
4) Hint: we need to construct two solutions of the homogeneous equation, using the exponential ansatz. Solution: The two independent solutions are e^{-x} and xe^{-x} (Notice that the indices of the characteristic equation coincide). The general solution is $y(x) = \frac{x^2 e^{-x}}{2} + (A_1 + A_2 x)e^{-x}$.
5) Hint: use the variation of constants method. Solution: $y(x) = Ae^{\frac{x^2}{2}} + \int_0^x e^{\frac{x^2 - s^2}{2}} ds$, where A is an arbitrary constant.

6) Hint: separable. Solution: $y(x) = \pm\frac{\sqrt{25-x^2}}{x^2}$.

7) Hint: use the potential method. Solution: the potential is (apart for additive constant) $U(x,y) = \frac{x^3}{3} + xe^y + y - 1$ and solutions are given in implicit form by $U(x,y) = A$.

8) Hint: (for instance) you can use the variation of constants method to find the particular solution to the inhomogeneous equation (but you can also probably guess it). Solution: $y(x) = 1 + 2e^{-\frac{x^3}{3}}$.

9) Hint: Construct a second independent solution by variation of constants. Solution: the second solution is obtained after solving the ODE with the ansatz $y_2(x) = a_2(x)x$. We find $y_2(x) = x\int^x e^{-s^2}\frac{ds}{s^2}$. The general solution is $y(x) = A_1x + A_2x\int^x e^{-s^2}\frac{ds}{s^2}$.

10) Hint: Potential method. Solution: $y(x) = \frac{x^2-\cos(x)+1}{\cos(x)+1}$.

0.2 Linear 2nd order equations. Fuchsian points and P-symbol method

Exercises

Ex 1. Discuss the possible singularities of the ODE

$$x^4y'' + y = 0, \tag{0.1}$$

and in particular the form of the solution at $x \sim 0$ and $x \sim \infty$.

Ex. 2 - Classification of singular points. Study the singular points of the following equations.

a)
$$x^2y'' + xy' + (x^2 - a^2)y = 0.$$

(Bessel equation) In this case, compute the form of the series expansion around $x = 0$. (This was done in class).

b)
$$(1 - x^2)y'' - 2xy' + a(a+1)y = 0.$$

(Legendre equation).

c)
$$xy'' + (1 + a - x)y' + by = 0$$

(Laguerre equation),

where a, $b \in \mathbb{C}$.

Ex. 3 Study the equation:

$$x^2(x-2)y'' + xy' - y = 0.$$

Notice that it has only 3 singularities, all Fuchsian. The singularities are: $0, 2, \infty$, with indices $(1, \frac{1}{2})$, $(0, \frac{1}{2})$, $(0, -1)$, respectively.

Use the P-symbol method to write a basis of solutions around $x = 2$.

Ex 4. Consider the example of this ODE:

$$x(x+1)y'' - (x-1)y' + y = 0 \tag{0.2}$$

- Discuss the possible singular points, and their type.
- Find explicitly the first two terms of two independent series solutions around $x = 0$.
- Write a solution around $x = 0$ with the P-symbol method and check the above result.

Ex 5. La soluzione di una certa equazione differenziale ordinaria è descritta dal simbolo di Papperitz-Riemann:

$$f(t) = P\left\{ t;\ \begin{matrix} 1 & 2 & \infty \\ -\frac{3}{4} & \rho & -\frac{1}{4} \\ 1 & \frac{1}{2}-\rho & \frac{1}{2} \end{matrix} \right\}, \quad \rho \in \mathbb{R}. \tag{0.3}$$

Si scrivano, in termini di funzioni speciali esplicite, due soluzioni indipendenti dell'equazione che abbiano uno sviluppo in serie della forma

$$f(t) = \sum_{n=1}^{\infty} c_n t^{-\alpha-n}, \tag{0.4}$$

(dove α è un parametro diverso per le due soluzioni) per t sufficientemente grande. NON è richiesto di calcolare coefficienti c_n, ma di scrivere le due soluzioni in forma compatta in termini di funzioni speciali.

Solutions

Ex. 1 $x = 0$ is an irregular singularity, so there is no series solution around $x = 0$ with finitely many negative terms . The solution will have an essential singularity and a Laurent series with infinitely many negative powers.

$x = \infty$ is a regular singular point. In fact, writing the equation as $y'' + p(x)y + q(x) = 0$, $q(x) \sim O(1/x^4)$, and $p(x) \sim 0/x$ at infinity. Since $0 \neq 2$, infinity is a Fuchsian singularity and not a regular point. Plugging in the equation x^α, for $x \to \infty$ we find $\alpha(\alpha-1) = 0$. Since we are expanding around infinity where the natural variable is $1/x$, the indices are then 0 and -1. Notice that this is a resonant case since they differ by an integer. The solution which has the standard form is the one which is subleading, in this case, the one with the index 0. Namely, the form of the solutions around infinity are*

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{-n}, \quad y_2(x) = A\log(x) y_1(x) + x\sum_{n=0}^{\infty} b_n x^{-n}, \tag{0.5}$$

where these are series converging in a region of the form $|x| > R$, and A a constant to be fixed. To determine A, we must plug the solution into the ODE at large x. We notice that $y_2(x) \sim A\log(x)(1 + a_1/x + a_2/x^2 + \dots) + b_0 x + b_1 + b_2/x + \dots$. Plugging this expansion into the ODE and matching orders at $x \to \infty$, we find that we must have $A = 0$. So we find simply $y_2(x) = x\sum_{n=0}^{\infty} b_n x^{-n}$, where we can assume $b_0 = 1$, $b_1 = 0$, and the other coefficients are fixed by recursion.

Ex. 2 (a) - Bessel equation In this case there are two singular points (for generic parameter a): $x = 0$ is a Fuchsian singularity, and $x = \infty$ is an irregular singularity.

The solution cannot be found around infinity with the series expansion method.

We can write the solution as a series of the form:

$$y(x) = x^{\rho} \sum_{n=0}^{\infty} c_n x^n, \tag{0.6}$$

and this series will converge everywhere except for $x = \infty$. Plugging this expansion in the ODE we find at leading order:

$$\left[(\rho - 1)\rho + \rho - a^2\right] c_0 = 0, \tag{0.7}$$

so $\rho = \pm a$.

The following orders give in general:

$$c_n \left[(a+n)(a+n-1) + (a+n) - a^2\right] + c_{n-2} = 0, \tag{0.8}$$

where $c_{-1} = c_{-2} = 0$. For $n = 0$, this is automatically satisfied. For $n = 1$, it gives $c_1 = 0$, the next orders give

$$c_n = -c_{n-2}\,\frac{1}{n(n+2a)}, \quad n \geq 2. \tag{0.9}$$

This implies that all odd coefficients are zero (since $c_1 = 0$):

$$c_{2k+1} = 0. \tag{0.10}$$

For the even ones, we can iterate the previous equation to lower the index until we find:

$$c_{2k} = (-1)^k c_0\,\frac{1}{(2k)!!\,[(2k+2a)(2k+2a-2)\dots(2a+2)]}. \tag{0.11}$$

*Note that in the second solution we can always make the choice to fix $b_1 = 0$. In fact, this coefficient can be set to zero by redefining $y_2(x) \to y_2(x) - b_1/a_0 y_1(x)$.

Noting that $(2k)!! \equiv (2k)(2k-2)\dots 4\cdot 2 = 2^k(k!)$, and that

$$(2k+2a)(2k+2a-2)\dots(2a+2) \; = 2^k\,(a+1)(a+2)\dots(a+k) = (a+1)_k = \frac{\Gamma(a+1+k)}{\Gamma(a+1)},$$

then we can write the solution as

$$y(x) \;=\; x^a\sum_{k=0} c_{2k}x^{2k} = x^a c_0 \sum_{k=0}(-1)^k\left(\frac{x}{2}\right)^{2k}\frac{1}{(k!)(a+1)_k} \tag{0.12}$$

$$\;=\; x^a c_0 \sum_{k=0}(-1)^k\left(\frac{x}{2}\right)^{2k}\frac{\Gamma(a+1)}{(k!)\Gamma(a+1+k)}. \tag{0.13}$$

This solution (normalised with $c_0 = \frac{1}{\Gamma(a+1)2^a}$) is denoted as

$$J_a(x) \equiv \sum_{k=0}(-1)^k\left(\frac{x}{2}\right)^{2k+a}\frac{1}{(k!)\Gamma(a+1+k)} \tag{0.14}$$

(Bessel function of the first kind), for generic $a \in \mathbb{C}$. If a is not integer, the two independent solutions as $J_{\pm a}(x)$. If $a \in \mathbb{N}$, then the solution $J_a(x)$ still has the same form. The other solution will in general also contain a log contribution and can be obtained as a limit of the situation with $a \notin \mathbb{N}$.

Notice also that we can recognise the form above as a special kind of generalised hypergeometric:

$$J_a(x) \propto \,{}_0F_1(\,;a+1;-\frac{x}{2}). \tag{0.15}$$

Ex. 2 (b) - Legendre

- $x = 1$ is a Fuchsian singularity with indices $\rho = 0, 0$. The solution could have the form:

$$y_1(x) = \sum_{n\geq 0} a_n(x-1)^n, \quad y_2(x) = \sum_{n\geq 0} b_n(x-1)^n + A\log(x-1)\,y_1(x). \tag{0.16}$$

  Since the indices are degenerate, in this case we have with certainty $A \neq 0$.

- $x = -1$ is a Fuchsian singularity with indices $\rho = 0, 0$. The solution could have the form:

$$y_1(x) = \sum_{n\geq 0} a_n(x+1)^n, \quad y_2(x) = \sum_{n\geq 0} b_n(x+1)^n + A\log(x+1)\,y_1(x). \tag{0.17}$$

  Since the indices are degenerate, in this case we have with certainty $A \neq 0$.

- $x = \infty$ is a Fuchsian singularity with indices $\rho = a+1, -a$. The solution could have the form:

$$y_1(x) = x^{-a-1}\sum_{n\geq 0} a_n x^{-n}, \quad y_2(x) = x^a\sum_{n\geq 0} b_n x^{-n}. \tag{0.18}$$

Ex. 2 (c) - Laguerre

- $x = \infty$ is an irregular singularity.
- $x = 0$ is a Fuchsian singularity with indices $\rho = 0, -a$. So the solution could have the form:

$$y_1(x) = \sum_{n \geq 0} a_n x^n, \quad y_2(x) = x^{-a} \sum_{n \geq 0} b_n x^n. \tag{0.19}$$

Ex. 3 The solution can be represented in the P-symbol notation as

$$y(x) = P \left\{ \begin{matrix} & 2 & 0 & \infty \\ x & 0 & 1 & -1 \\ & \frac{1}{2} & \frac{1}{2} & 0 \end{matrix} \right\}. \tag{0.20}$$

Since we are interested in behaviour around $x = 2$, first we map the points $z_1 = 2$, $z_2 = 0$, $z_3 = \infty$ to the canonical positions 0,1,∞. This is done with the fractional linear transformation:

$$x \to \frac{2 - x}{2}, \tag{0.21}$$

so we have

$$y(x) = P \left\{ \begin{matrix} & 0 & 1 & \infty \\ \frac{2-x}{2}, & 0 & 1 & -1 \\ & \frac{1}{2} & \frac{1}{2} & 0 \end{matrix} \right\}. \tag{0.22}$$

Now we want to bring it to canonical form, i.e. we must have one zero index in column 1 and one in column 2. We are not there yet. So we use another property to redefine the indices:

$$y(x) = \left(\frac{2 - x}{2} - 1 \right) P \left\{ \begin{matrix} & 0 & 1 & \infty \\ \frac{2-x}{2}, & 0 & 0 & 0 \\ & \frac{1}{2} & -\frac{1}{2} & 1 \end{matrix} \right\}. \tag{0.23}$$

(Notice how we wanted to change the indices for the point 1, but they get automatically redefined also at infinity!)

This is now in canonical form, so from here we can read one solution:[†‡]

$$y_1(x) = \left(-\frac{x}{2} \right) {}_2F_1(0, 1; 1/2; 1 - \frac{x}{2}) = -\frac{x}{2}. \tag{0.24}$$

We denoted this as y_1 since it has the leading behaviour at $x \sim 2$.

[†] Check the relevant formula in the notes or in the "UsefulEquations" file.

[‡] (Note: actually in this case, since $(0)_i = \delta_{i,0}$, the hypergeometric reduces to 1 and the solution is simply $\propto x$. This is just a coincidence of the data of the problem).

The second independent solution with the leading behaviour $(x-2)^{\frac{1}{2}}$ is found by starting with the equation written as (i.e., we swap the two indices for z_1).

$$y(x) = P\left\{ \begin{array}{cccc} & 0 & 1 & \infty \\ \frac{2-x}{2}, & \frac{1}{2} & 1 & -1 \\ & 0 & \frac{1}{2} & 0 \end{array} \right\}. \tag{0.25}$$

Then, using the property to redefine the indices:

$$y(x) = \left(\frac{2-x}{2}\right)^{\frac{1}{2}} \left(\frac{2-x}{2} - 1\right) P\left\{ \begin{array}{cccc} & 0 & 1 & \infty \\ \frac{2-x}{2}, & 0 & 0 & \frac{1}{2} \\ & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{array} \right\}. \tag{0.26}$$

From this canonical form we read the other independent solution:[§]

$$y_2(x) = \left(\frac{2-x}{2}\right)^{\frac{1}{2}} \left(\frac{2-x}{2} - 1\right) {}_2F_1(\frac{1}{2}, \frac{3}{2}; \frac{3}{2}; \frac{2-x}{2}). \tag{0.27}$$

Ex. 4

- The singularities are $z_1 \equiv 0$, $z_2 \equiv -1$, $z_3 \equiv \infty$. We can check that they are all Fuchsian.

  The behaviour of solutions around z_1 $(x = 0)$ is $y \sim x^{\alpha}$, plugging in the ODE we find the indicial equation $\alpha^2 = 0$.

  Solutions around z_2 $(x = -1)$ behave like $y \sim (x+1)^{\alpha}$, plugging in the ODE we find the indicial equation $\alpha(\alpha - 3) = 0$ (which is again a resonant case since they differ by an integer).

  Around z_3 (infinity), taking the form $y \sim x^{-\alpha}$ and expanding the ODE for $x \to \infty$ we find $\alpha^2 + 2\alpha + 1 = 0$, so the indices are both -1.

- Around $x = 0$, since the indices are 0 and 0 (degenerate case), we can take two solutions of the form: $y_1(x) = a_0 + a_1 x + \dots$ and $y_2(x) = (b_0 + b_1 x + \dots) + \log(x) A(a_0 + a_1 x + \dots)$. We can normalize $a_0 = 1$ without loss of generality.

  Plugging these expansions in the ODE, at the leading order from the equation for y_1 we find $a_0 + a_1 = 0$, so we can take $y_1(x) = 1 - x + O(x^2)$.

  From the equation for $y_2(x)$ we then find, at leading order, $4A - b_0 - b_1 = 0$. Subtracting a quantity proportional to y_1, we can assume without loss of generality $b_0 = 0$. Therefore, we find $y_2(x) = A \log x \left(1 - x + O(x^2)\right) + 4Ax + O(x^2)$ (we can set $A = 1$ for normalisation).

[§]In this case it turns out this is a simple algebraic function (again, a coincidence of the data of the problem which produce a "simple" hypergeometric function).

- With the P-symbol notation the generic solution is written as

$$y(x) = P \left\{ \begin{matrix} & 0 & -1 & \infty \\ x & 0 & 0 & -1 \\ & 0 & 3 & -1 \end{matrix} \right\} \tag{0.28}$$

(Notice the subtlety that the solution behaves like $\sim x^1$ at $x \sim \infty$, but this corresponds to index -1, not $+1$, because the "distance" from infinity is $1/x$, not x).

Using the fractional linear transformation $x \to x' = -x$, we rewrite this in the canonical form:

$$y(x) = P \left\{ \begin{matrix} & 0 & 1 & \infty \\ -x & 0 & 0 & -1 \\ & 0 & 3 & -1 \end{matrix} \right\}. \tag{0.29}$$

This is in canonical form with $c = 1$, $a = b = -1$, so we can read one solution immediately:

$$y(x) =_2 F_1(-1, -1; 1; -x). \tag{0.30}$$

This is given by a power series around $x = 0$, starting as $1 - x + \dots$, so indeed it agrees with one of the solutions above.¶

It is a bit trickier to describe the second solution with the P-symbol method because of the degeneracy, although it can also be done by introducing regularising parameters.

Ex. 5 Il P-symbol mostra che ci sono due soluzioni indipendenti caratterizzate da due andamenti diversi intorno a infinito:

$$f_1(t) = t^{\frac{1}{4}-n} c_n, \quad f_2(t) = t^{-\frac{1}{2}-n} d_n, \tag{0.31}$$

dove gli indici sono quelli indicati dal P-symbol. Ogni serie converge per t sufficientemente grande. Tali soluzioni (ognuna definita a meno di una costante arbitraria) sono chiaramente indipendenti, e si possono scrivere esplicitamente in termini di funzioni ipergeometriche. Per far questo occorre manipolare il P-symbol, ci sono diversi possibili modi di farlo, il seguito mostra un possibile svolgimento che porta a uno dei modi di esprimere il risultato.

Introduco una mappa lineare fratta che mandi $1 \to \infty$, $2 \to 1$, $\infty \to 0$. La mappa è:

$$t \to t' = \frac{1}{t-1}. \tag{0.32}$$

Il P-simbolo diventa

$$f(t) = P \left\{ t'; \begin{matrix} 0 & 1 & \infty \\ -\frac{1}{4} & \rho & -\frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} - \rho & 1 \end{matrix} \right\}, \tag{0.33}$$

¶Note: in fact, since $a \in \mathbb{Z}_{<0}$, the solution truncates and it is just a polynomial $y_1(x) = 1 - x$.

e con alcuni passaggi standard arrivo a una forma canonica:

$$f(t) = (t')^{-\frac{1}{4}}(t'-1)^{\rho} P \begin{Bmatrix} & 0 & 1 & \infty \\ t'; & 0 & 0 & -\frac{3}{4}-\frac{1}{4}+\rho \\ & \frac{3}{4} & \frac{1}{2}-2\rho & 1-\frac{1}{4}+\rho \end{Bmatrix}, \tag{0.34}$$

da cui posso estrarre nel modo standard una delle due soluzioni, quella con l'andamento $t^{\frac{1}{4}}$ per grande t, che corrisponde a $(t')^{-\frac{1}{4}}$ per $t' \to 0$. Questa soluzione è (una volta sostituito $t' \to 1/(t-1)$):

$$f_1(t) \propto (t-1)^{+\frac{1}{4}}(\frac{2-t}{t-1})^{\rho}\ {}_2F_1(\rho-1, \rho+\frac{3}{4}; \frac{1}{4}; \frac{1}{t-1}). \tag{0.35}$$

Per trovare la seconda soluzione, $f_2(t)$, torno a (0.33) ma estraggo un andamento diverso per $t' \to 0$:

$$f(t) = (t')^{\frac{1}{2}}(t'-1)^{\rho} P \begin{Bmatrix} & 0 & 1 & \infty \\ t'; & -\frac{1}{4}-\frac{1}{2} & 0 & -\frac{3}{4}+\frac{1}{2}+\rho \\ & 0 & \frac{1}{2}-2\rho & 1+\frac{1}{2}+\rho, \end{Bmatrix}, \tag{0.36}$$

da cui leggo la seconda soluzione indipendente. Una volta sostituito di nuovo $t' = 1/(t-1)$:

$$f_2(t) \propto (t-1)^{-\frac{1}{2}}(\frac{2-t}{t-1})^{\rho}\ {}_2F_1(\rho-\frac{1}{4}, \rho+\frac{3}{2}; \frac{7}{4}; \frac{1}{t-1}). \tag{0.37}$$

Si noti che usando la definizione come serie delle funzioni ipergeometriche, da queste formule si possono facilmente leggere i coefficienti di un'espansione in serie in potenze di $\frac{1}{t-1}$. Questa serie converge per $|\frac{1}{t-1}| < 1$, che corrisponde a una regione per grande t. La serie si potrebbe riespandere per ottenere espansioni della forma (0.31), il che comunque non è richiesto nell'esercizio.