Es. 2 EQ. DI LEGENDRE GENERALIZZATA

$$(1-x^2)\,y'' - 2x\,y' + \left(\lambda(\lambda+1) - \frac{\mu^2}{1-x^2}\right) y = 0$$

- I PUNTI SINGOLARI SONO $x=+1$, $x=-1$ E $x=\infty$.

SONO TUTTI FUCHSIANI.

- $x=+1$.

$$p_0 = \lim_{x\to 1} (x-1)\,p(x) = \lim_{x\to 1} \frac{2x}{x+1}\frac{\cancel{(x-1)}}{\cancel{(x-1)}} = 1$$

$$q_0 = \lim_{x\to 1} (x-1)^2\,q(x) = \lim_{x\to 1}\left(-\frac{(x-1)^2}{(1-x^2)^2}\mu^2\right) = -\frac{\mu^2}{4}$$

INDICES: $\rho(\rho-1) + 1\cdot\rho - \frac{\mu^2}{4} = \rho^2 - \frac{\mu^2}{4} = 0$

$$\hookrightarrow \rho = \pm\frac{\mu}{2}$$

- $x=-1$

$$p_0 = \lim_{x\to -1} (x+1)\,p(x) = \lim_{x\to -1} \frac{(x+1)}{1-x^2}(-2x) = 1$$

$$q_0 = \lim_{x\to -1} (x+1)^2\,q(x) = \lim_{x\to -1} (x+1)^2\frac{(-\mu^2)}{(1-x^2)^2} = -\frac{\mu^2}{4}$$

$\rightarrow$ INDICES : $\rho = \pm\frac{\mu}{2}$

- $x=\infty$. USING THE INDICIAL EQ. FOR EXPANDING AROUND ∞:

$$\rho(\rho+1) - \tilde{p}_0\,\rho - \tilde{q}_0 = 0 \quad ,$$

WITH $\tilde{p}_0 = \lim_{x \to \infty} x\, p(x) = \lim_{x \to \infty} \frac{(-2x^2)}{(1-x^2)} = 2$

$$\tilde{q}_0 = \lim_{x\to\infty} x^2\, q(x) = \lim_{x\to\infty} \frac{\lambda(\lambda+1)}{(1-x^2)}\, x^2 = -\lambda(\lambda+1)$$

SO THE INDICES ARE: $\rho^2 - \rho - \lambda(\lambda+1) = 0$

$$\hookrightarrow \quad \rho = -\lambda, \text{ OR } \rho = \lambda + 1.$$

THE SUM OF INDICES IS $(\lambda+1) - \lambda \pm \frac{\mu}{2} \pm \frac{\mu}{2} = +1$ ✓

- THE SOLUTIONS CAN BE REPRESENTED AS:

$$y(x) = P\left\{\begin{matrix} & 1 & -1 & \infty \\ x & \frac{\mu}{2} & \frac{\mu}{2} & \lambda+1 \\ & -\frac{\mu}{2} & -\frac{\mu}{2} & -\lambda \end{matrix}\right\}$$

WE CAN CONSTRUCT A BASIS OF TWO IND. SOLUTIONS WITH BEHAVIOURS $y \sim (x-1)^{\frac{\mu}{2}}$

OR $y \sim (x-1)^{-\frac{\mu}{2}}$

(WE ASSUME HERE THAT $\mu \neq 0$ AND $\mu \notin \mathbb{Z}$).

TO FIND THE SOLUTION WITH BEHAVIOUR $\sim (x-1)^{\frac{\mu}{2}}$,

WE FIRST FIND A TRANSFORMATION SENDING

(THIS IS A POSSIBLE METHOD, THE EXERCISE CAN BE SOLVED IN OTHER WAYS TOO! ONE WOULD FIND SOLUTIONS THAT "LOOK" DIFFERENT BUT ARE EQUIVALENT -)

$$\begin{array}{rcl} 1 & \to & 0 \\ -1 & \to & 1 \\ \infty & \to & \infty \end{array}$$

$\downarrow$

THIS MAP IS GIVEN BY $x \longrightarrow \tilde{x} = \dfrac{1-x}{2}$.

THEN WE HAVE:

$$y = P\left\{ \begin{array}{cccc} & 0 & 1 & \infty \\ \tilde{x} & \frac{\mu}{2} & \mu/2 & \lambda+1 \\ & -\frac{\mu}{2} & -\frac{\mu}{2} & -\lambda \end{array} \right\}$$

USING ANOTHER TRANSFORMATION WE GET:

$$y = \underbrace{(\tilde{x})^{\frac{\mu}{2}}} \cdot P\left\{ \begin{array}{cccc} & 0 & 1 & \infty \\ \tilde{x} & 0 & \frac{\mu}{2} & \lambda+1+\frac{\mu}{2} \\ & -\mu & -\frac{\mu}{2} & -\lambda+\frac{\mu}{2} \end{array} \right\} \qquad (*)$$

WE ISOLATED THIS FACTOR SINCE IT GOES LIKE $(x-1)^{\frac{\mu}{2}}$ FOR $x \to 1$ (SINCE $\tilde{x} \propto 1-x$)

FINALLY, WE USE ANOTHER TRANSF. TO GET TO A CANONICAL FORM:

$$y = (\tilde{x})^{\frac{\mu}{2}} \cdot (\tilde{x} - 1)^{-\frac{\mu}{2}} \cdot P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & \mu & \lambda+1 & \tilde{x} \\ -\mu & 0 & -\lambda & \end{Bmatrix}$$

FROM THIS WE CAN EXTRACT A SOLUTION:

$$\boxed{y_1(x) = \left(\frac{1-x}{-x-1}\right)^{\frac{\mu}{2}} {}_2F_1\left(\lambda+1, -\lambda ; \mu+1 ; \frac{1-x}{2}\right)}$$

↳ WHICH INDEED BEHAVES AS $\sim (1-x)^{\frac{\mu}{2}}$

FOR $x \to 1$.

SIDE NOTE: DESPITE THIS FACTOR, WE CANNOT DRAW CONCLUSIONS FROM THIS EQUATION ON THE BEHAVIOUR OF THIS SOLUTION FOR $x \to -1$.

THIS IS BECAUSE THE HYPERGEOMETRIC series CONVERGES FOR $\left|\frac{1-x}{2}\right| < 1$, WHICH MEANS THAT AS $x \to -1$ THE SERIES IS NOT CONVERGING.

• IF $\mu \in \mathbb{Z}$, WE CAN FIND EASILY A SECOND INDEPENDENT SOLUTION WITH DIFFERENT BEHAVIOUR $\sim (x-1)^{-\frac{\mu}{2}}$ FOR $x \to 1$.

SINCE THE ORIGINAL EQ. (AND THE P-SYMBOL) ARE INVARIANT FOR $\mu \leftrightarrow -\mu$, THIS SECOND SOLUTION IS SIMPLY GIVEN BY:

$$\boxed{\begin{aligned} y_2(x) &= \Big(y_1(x) \text{ WITH } (\mu \leftrightarrow -\mu) \Big) \\ &= \left(\frac{x-1}{x+1}\right)^{-\frac{\mu}{2}} \cdot {}_2F_1\left(\lambda+1, -\lambda; 1-\mu; \frac{1-x}{2}\right). \end{aligned}}$$

SO THE BASIS OF INDEPENDENT SOLUTIONS IS GIVEN BY $y_1(x)$, $y_2(x)$ ABOVE.

• TO SEE THAT THE SOLUTION WITH BEHAVIOUR $\underset{x\to 1}{\sim} (x-1)^{\frac{\mu}{2}}$ IS A POLYNOMIAL WHEN $\frac{\mu}{2} \in \mathbb{N}$, $\lambda \in \mathbb{N}$, $\lambda - \mu \in \mathbb{N}$, LET US GO

BACK TO THE P- SYMBOL, SINCE IT IS MORE CONVENIENT TO REWRITE $y_1(x)$ IN A DIFFERENT FORM.

WE ALREADY FOUND ((*) ABOVE):

$$y = (\tilde{x})^{\frac{\mu}{2}} \cdot P \left\{ \begin{matrix} & 0 & 1 & \infty \\ \tilde{x} & 0 & \frac{\mu}{2} & \lambda+1+\frac{\mu}{2} \\ & -\mu & -\frac{\mu}{2} & -\lambda+\frac{\mu}{2} \end{matrix} \right\}$$

NOW LET US TAKE AN ALTERNATIVE (BUT EQUIVALENT!) ROUTE TO GET TO THE CANONICAL FORM:

$$\hookrightarrow y = (\tilde{x})^{\frac{\mu}{2}} (\tilde{x}-1)^{\frac{\mu}{2}} \cdot P \left\{ \begin{matrix} & 0 & 1 & \infty \\ \tilde{x} & 0 & 0 & \lambda+1+\mu \\ & -\mu & -\mu & -\lambda+\mu \end{matrix} \right\}$$

FROM THIS WE EXTRACT

$$\tilde{y}_1 = (1-x)^{\frac{\mu}{2}} (1+x)^{\frac{\mu}{2}} \cdot {}_2F_1\left(\lambda+1+\mu, \underbrace{-\lambda+\mu}_{\in \mathbb{Z}_{\leq 0}}; \mu+1; \frac{1-x}{2}\right).$$

THIS IS A POLYNOMIAL FOR $\frac{\mu}{2} \in \mathbb{N}$, $\lambda - \mu \in \mathbb{N}$!

IN FACT, WHEN $\lambda - \mu \in \mathbb{N}$, WE HAVE A ${}_2F_1$ FUNCTION WITH A COEFFICIENT (IN THE NUMERATOR) GIVEN BY A NON-POSITIVE INTEGER: AS WE SAW, THIS MEANS THAT THE SERIES EXPANSION FOR A GEN. HYPERGEOMETRIC FUNCTION TRUNCATES AFTER A FINITE N. OF TERMS.

NOTE: NOTICE THAT $\tilde{y}_1$ MUST BE JUST PROPORTIONAL TO y_1, WHEN $\mu \geq 0$, SINCE THEY HAVE THE SAME SUBLEADING BEHAVIOUR FOR $x \to 1$. THE TWO EXPRESSIONS FOUND ABOVE FOR y_1 AND $\tilde{y}_1$ ARE JUST EQUIVALENT, APART FOR A PROPORTIONALITY CONSTANT.

Above, “subleading” means that these solutions have a behaviour characterised by the index giving the behaviour that is “smaller” (namely, the one corresponding to the bigger of the two indices). Notice that the subleading solution is always unique apart for a multiplicative constant.