Exercises on Ordinary Differential Equations

0.1 Direct methods

Write the solutions to the following ODE problems for the function $y(x)$ using the methods discussed in the lectures. The methods are: separation of variables, the potential method, and methods for homogeneous and inhomogeneous linear equations, in particular variation of constants.

Problems

1) Find the general solution to $xy'(x) = 1 + y(x)$;

2) Find the solution to $xy' - xy = y$ such that $y = 1$ for $x = 1$;

3) Write the general solution to $y'\left(y + x^2\right) + 2xy + \sin(x) = 0$.

4) Consider the ODE $y'' + 2y' + y = e^{-x}$. A particular solution is given by $y = \frac{x^2 e^{-x}}{2}$. Write the general solution.

5) Find the general solution to $y' - xy = 1$.

6) Solve $(1 + y^2) + xyy' = 0$ with $y = 0$ when $x = 5$.

7) Write in implicit form the general solution to $y'(xe^y + 1) + x^2 + e^y = 0$.

8) Find the solution to $y' + x^2 y = x^2$ such that $y = 3$ for $x = 0$.

9) Consider the ODE: $y'' + 2xy' - 2y = 0$. A solution is given by $y(x) = x$. Write the general solution.

10) Solve $(\cos(x) + 1)y' - (y + 1)\sin(x) - 2x = 0$, with $y(0) = 0$.

Solutions and hints:

1) Hint: separable equation. Solution: $y(x) = Ax - 1$.

2) Hint: separable. Solution: $y(x) = e^{-1+x}x$.

3) Hint: use the potential method. Solution: $y(x) = \pm\sqrt{A + x^4 + 2\cos(x) - 1} - x^2$.

4) Hint: we need to construct two solutions of the homogeneous equation, using the exponential ansatz. Solution: The two independent solutions are e^{-x} and xe^{-x} (Notice that the indices of the characteristic equation coincide). The general solution is $y(x) = \frac{x^2 e^{-x}}{2} + (A_1 + A_2 x)e^{-x}$.

5) Hint: use the variation of constants method. Solution: $y(x) = Ae^{\frac{x^2}{2}} + \int_0^x e^{\frac{x^2 - s^2}{2}} ds$, where A is an arbitrary constant.

6) Hint: separable. Solution: $y(x) = \pm\frac{\sqrt{25-x^2}}{x^2}$.

7) Hint: use the potential method. Solution: the potential is (apart for additive constant) $U(x,y) = \frac{x^3}{3} + xe^y + y - 1$ and solutions are given in implicit form by $U(x,y) = A$.

8) Hint: (for instance) you can use the variation of constants method to find the particular solution to the inhomogeneous equation (but you can also probably guess it). Solution: $y(x) = 1 + 2e^{-\frac{x^3}{3}}$.

9) Hint: Construct a second independent solution by variation of constants. Solution: the second solution is obtained after solving the ODE with the ansatz $y_2(x) = a_2(x)x$. We find $y_2(x) = x\int^x e^{-s^2}\frac{ds}{s^2}$. The general solution is $y(x) = A_1x + A_2x\int^x e^{-s^2}\frac{ds}{s^2}$.

10) Hint: Potential method. Solution: $y(x) = \frac{x^2-\cos(x)+1}{\cos(x)+1}$.

0.2 Linear 2nd order equations. Classification of singular points, first part.

Exercises

Ex 1. Discuss the possible singularities of the ODE

$$x^4y'' + y = 0, \tag{0.1}$$

and in particular the form of the solution at $x \sim 0$ and $x \sim \infty$.

Ex. 2 - Classification of singular points. Study the singular points of the following equations.

a)

$$x^2y'' + xy' + (x^2 - a^2)y = 0.$$

(Bessel equation) In this case, compute the form of the series expansion around $x = 0$, generalizing the method to compute the expansion of the Airy equations seen in class.

b)

$$(1-x^2)y'' - 2xy' + a(a+1)y = 0.$$

(Legendre equation).

c)

$$xy'' + (1+a-x)y' + by = 0$$

(Laguerre equation),

where a, $b \in \mathbb{C}$.

Solutions

Ex. 1 $x = 0$ is an irregular singularity, so there is no series solution around $x = 0$ with finitely many negative terms . The solution will have an essential singularity and a Laurent series with infinitely many negative powers.

$x = \infty$ is a regular singular point. In fact, writing the equation as $y'' + p(x)y + q(x) = 0$, $q(x) \sim O(1/x^4)$, and $p(x) \sim 0/x$ at infinity. Since $0 \neq 2$, infinity is a Fuchsian singularity and not a regular point. Plugging in the equation x^α, for $x \to \infty$ we find $\alpha(\alpha-1) = 0$. Since we are expanding around infinity where the natural variable is $1/x$, the indices are then 0 and -1. Notice that this is a resonant case since they differ by an integer. The solution which has the standard form is the one which is subleading, in this case, the one with the index 0. Namely, the form of the solutions around infinity are*

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{-n}, \quad y_2(x) = A\log(x) y_1(x) + x\sum_{n=0}^{\infty} b_n x^{-n}, \tag{0.2}$$

where these are series converging in a region of the form $|x| > R$, and A a constant to be fixed. To determine A, we must plug the solution into the ODE at large x. We notice that $y_2(x) \sim A\log(x)(1 + a_1/x + a_2/x^2 + \dots) + b_0 x + b_1 + b_2/x + \dots$. Plugging this expansion into the ODE and matching orders at $x \to \infty$, we find that we must have $A = 0$. So we find simply $y_2(x) = x\sum_{n=0}^{\infty} b_n x^{-n}$, where we can assume $b_0 = 1$, $b_1 = 0$, and the other coefficients are fixed by recursion.

Ex. 2 (a) - Bessel equation In this case there are two singular points (for generic parameter a): $x = 0$ is a Fuchsian singularity, and $x = \infty$ is an irregular singularity.

The solution cannot be found around infinity with the series expansion method.

We can write the solution as a series of the form:

$$y(x) = x^\rho \sum_{n=0}^{\infty} c_n x^n, \tag{0.3}$$

and this series will converge everywhere except for $x = \infty$. Plugging this expansion in the ODE we find at leading order:

$$\left[(\rho-1)\rho + \rho - a^2\right] c_0 = 0, \tag{0.4}$$

so $\rho = \pm a$.

The following orders give in general:

$$c_n\left[(a+n)(a+n-1) + (a+n) - a^2\right] + c_{n-2} = 0, \tag{0.5}$$

*Note that in the second solution we can always make the choice to fix $b_1 = 0$. In fact, this coefficient can be set to zero by redefining $y_2(x) \to y_2(x) - b_1/a_0 y_1(x)$.

where $c_{-1} = c_{-2} = 0$. For $n = 0$, this is automatically satisfied. For $n = 1$, it gives $c_1 = 0$, the next orders give

$$c_n = -c_{n-2}\,\frac{1}{n(n+2a)}, \quad n \geq 2. \tag{0.6}$$

This implies that all odd coefficients are zero (since $c_1 = 0$):

$$c_{2k+1} = 0. \tag{0.7}$$

For the even ones, we can iterate the previous equation to lower the index until we find:

$$c_{2k} = (-1)^k c_0\,\frac{1}{(2k)!!\,[(2k+2a)(2k+2a-2)\dots(2a+2)]}. \tag{0.8}$$

Noting that $(2k)!! \equiv (2k)(2k-2)\dots 4\cdot 2 = 2^k(k!)$, and that

$$(2k+2a)(2k+2a-2)\dots(2a+2) \;= 2^k\,(a+1)(a+2)\dots(a+k) = (a+1)_k = \frac{\Gamma(a+1+k)}{\Gamma(a+1)},$$

then we can write the solution as

$$\begin{aligned} y(x) &= x^a \sum_{k=0} c_{2k} x^{2k} = x^a c_0 \sum_{k=0} (-1)^k \left(\frac{x}{2}\right)^{2k} \frac{1}{(k!)(a+1)_k} && (0.9)\\ &= x^a c_0 \sum_{k=0} (-1)^k \left(\frac{x}{2}\right)^{2k} \frac{\Gamma(a+1)}{(k!)\Gamma(a+1+k)}. && (0.10) \end{aligned}$$

This solution (normalised with $c_0 = \frac{1}{\Gamma(a+1)2^a}$) is denoted as

$$J_a(x) \equiv \sum_{k=0} (-1)^k \left(\frac{x}{2}\right)^{2k+a} \frac{1}{(k!)\Gamma(a+1+k)} \tag{0.11}$$

(Bessel function of the first kind), for generic $a \in \mathbb{C}$. If a is not integer, the two independent solutions as $J_{\pm a}(x)$. If $a \in \mathbb{N}$, then the solution $J_a(x)$ still has the same form. The other solution will in general also contain a log contribution and can be obtained as a limit of the situation with $a \notin \mathbb{N}$.

Notice also that we can recognise the form above as a special kind of generalised hypergeometric:

$$J_a(x) \propto {}_0F_1(\,;a+1;-\frac{x}{2}). \tag{0.12}$$

Ex. 2 (b) - Legendre

- $x = 1$ is a Fuchsian singularity with indices $\rho = 0, 0$. The solution could have the form:

$$y_1(x) = \sum_{n\geq 0} a_n (x-1)^n, \quad y_2(x) = \sum_{n\geq 0} b_n (x-1)^n + A \log(x-1)\, y_1(x). \tag{0.13}$$

- $x = -1$ is a Fuchsian singularity with indices $\rho = 0, 0$. The solution could have the form:

$$y_1(x) = \sum_{n \geq 0} a_n (x+1)^n, \quad y_2(x) = \sum_{n \geq 0} b_n (x+1)^n + A \log{(x+1)}\, y_1(x). \tag{0.14}$$

- $x = \infty$ is a Fuchsian singularity with indices $\rho = a+1, -a$. The solution could have the form:

$$y_1(x) = x^{-a-1} \sum_{n \geq 0} a_n x^{-n}, \quad y_2(x) = x^{a} \sum_{n \geq 0} b_n x^{-n}. \tag{0.15}$$

Ex. 2 (c) - Laguerre

- $x = \infty$ is an irregular singularity.
- $x = 0$ is a Fuchsian singularity with indices $\rho = 0, -a$. So the solution could have the form:

$$y_1(x) = \sum_{n \geq 0} a_n x^n, \quad y_2(x) = x^{-a} \sum_{n \geq 0} b_n x^n. \tag{0.16}$$