1 Exercises on 2nd order PDEs

Note: In all these exercises, Fourier coefficients can be expressed as integrals. It is not required to compute the integrals explicitly, except when required.

1.1 Heat equation

Hint: When the problem has non-homogeneous boundary conditions, it is convenient to subtract an appropriate function in order to get a problem with homogeneous boundary conditions.

Ex. 0

Si consideri l'evoluzione della temperatura $u(x,t)$ di una barretta metallica di lunghezza L, in presenza di un processo esterno caratterizzato da una funzione $F(x,t) = A\cos(t)\sin(\pi\frac{x}{L})$ che pompa calore nel sistema:

$$u_t - \alpha\, u_{xx} = A\cos(t)\sin(\pi\frac{x}{L}), \quad 0 \le x \le L, \tag{1.1}$$

$\alpha > 0$, dove t denota il tempo. Le condizioni iniziali e ai bordi siano

$$u(0,t) = u(x,L) = 0, \;\; u(x,0) = x^2(x-L)^2. \tag{1.2}$$

- Si risolva il problema nel caso particolare $A = 0$.
- Si trovi una soluzione particolare $v(x,t)$ tale che $u(x,t) = u_{A=0}(x,t) + v(x,t)$ soddisfi il problema completo con $A \neq 0$, dove $u_{A=0}(x,t)$ è la soluzione trovata al punto precedente.

Ex. 1 (a)

Consider a bar of length L with heat diffusivity coefficient α. The bar's ends are kept at constant temperatures $u(0,t) = T_1$, $u(L,t) = T_2$. The initial temperature is $u(x,t=0) = 4(L-x)^2x^2$.

- What is the temperature distribution at $t = \infty$? (This can be answered without explicit calculations).
- Write the solution at generic times.

Ex. 1 (b)

Consider the same setup as above, with initial condition a constant $u(x,t=0) = T_0$, and the two ends both kept at constant temperature $T = 0$, i.e. $u(0,t) = u(L,t) = 0$. Write the solution at generic times as a Fourier series.

NOTE: this was discussed in lecture 19.

Ex. 2 - One-dimensional, different boundary conditions

A bar of length $L = \pi$ is initially at temperature $T = 0$ at time $t = 0$. At times $t > 0$, the left end of the bar is insulated, while the right end is kept at constant temperature $T = 50$. Find an expression for the time-dependent temperature distribution.

Hint: The fact that the left end is insulated means that there is no heat flow across this end. Remember that the heat flow is proportional to the gradient of the temperature.

Ex. 3 - With time-dependent boundary conditions

At time $t = 0$, a bar of length L has uniform temperature $u(x, t = 0) = 0$, $0 \leq x \leq L$. For $t > 0$, the endpoints of the bar are heated such that

$$u(x = 0, t) = 3t, \quad u(x = L, t) = 4t, \tag{1.3}$$

for $0 \leq t \leq 1$. What is the temperature distribution at $t = 1$?

Hint: Do a subtraction in order to get a homogeneous boundary condition. Notice that in this case this will lead you to a non-homogeneous PDE to solve. You can solve it using a Fourier decomposition with time-dependent coefficients (the same trick that we used to solve the problem of the wave equation subject to an external force). Derive the ODE satisfied by the coefficients and solve it.

Ex. 4 - Neumann problem

At time $t = 0$, a bar of length L has temperature distribution $u(x, t = 0) = x(1-x)$, $0 \leq x \leq L$. For $t > 0$, the endpoints of the bar are kept thermally insulated.

What is the temperature distribution at generic times $t > 0$? And what is the temperature at $t \to +\infty$? Show that this value is consistent with the conservation of thermal energy.

NOTE: this was discussed in lecture 19.

Ex. 5

Si consideri l'evoluzione della temperatura $u(x, t)$ di una barretta posta lungo l'intervallo $x \in [0, L]$ (con coefficiente di diffusività termica $\alpha = 1$):

$$u_t - u_{xx} = 0, \quad t \geq 0, \; x \in [0, L]. \tag{1.4}$$

La temperatura è uniforme, $u(x, t) = 0$ per $t < 0$. Al tempo $t = 0$, il centro della barretta viene scaldato generando una condizione iniziale data da una delta di Dirac:

$$u(x, t) = \delta(x - \frac{L}{2})T_0, \text{ con } T_0 > 0. \tag{1.5}$$

- (10 punti) Supponendo che i bordi della barretta siano mantenuti **termicamente isolati**, si determini $u(x, t)$ per $t > 0$ con il metodo di Fourier.

- (1 punto) Si calcoli la temperatura assunta dalla barretta dopo un tempo infinito $t \to +\infty$, mostrando come questo valore sia consistente con la legge di conservazione dell'energia termica. Nota: la densità di energia termica è proporzionale alla temperatura.

Suggerimento: Si usi il metodo di decomposizione di Fourier. Data la semplicità delle condizioni iniziali, è possibile calcolare esplicitamente i coefficienti a partire dalla loro definizione come integrali.

Ex. 6 - Problem with time dependent boundary conditions

Consider the heat propagation inside a 1D medium (for $x \in [0, L]$), which is initially at temperature $u(x, t = 0) = 0$, $0 \leq x \leq L$.
For $t > 0$, the left endpoint is kept at temperature $u(0, t) = 0$, while the right endpoint is heated with a certain time dependence, such that its temeperature is given by $u(L, t) = f(t)$ (with $f(t)$ a certain function of time).

What is the temperature distribution at generic times $t > 0$?

NOTE: this was discussed in lecture 19. To solve this problem it is necessary to subtract an explicit function, to reduce to a problem with homogeneous boundary conditions, but a inhomogeneous equation. This inhomogeneous equation can then be solved using the method discussed. Formula (1.25) might be useful.

Ex. 7 - Cooling of a sphere

Write the equations describing the cooling of a sphere of radius R, with initial uniform temperature $u(\vec{x}, t = 0) = T_0 > 0$, which is immersed in a space with uniform temperature $T_{\text{ext}} = 0$.

Hint: use spherical coordinates. See formulas at the end of the file for the relevant eigenfunctions (there are some simplifications given this initial distribution).

1.2 Wave equation

Ex. 5 - 1D

A string of length L, fixed at the points $x = 0$ and $x = L$ on the horizontal axis, has initially a displacement of the form $u(x, t = 0) = \sin^2(\frac{x\pi}{L})$, and its transversal velocity is $\frac{\partial}{\partial t} u(x, t)\big|_{t=0} = 0$. The endpoints of the strings are kept fixed. The propagation speed for waves on the string is v.

- Is the motion of the string for $t > 0$ periodic? If yes, what is its period (i.e., minimal time such that the motion repeats itself)?

- Derive an explicit expression for $u(x, t)$ for $t > 0$ (it can be written as an infinite Fourier series, which does not need to be summed. The coefficients should be defined explicitly, as integrals).

Hint: The first point can be answered both using the method of images and using the Fourier decomposition method. Try to deduce the answer using both methods.

Una grandezza $u(x,t)$ è descritta dall'equazione delle onde sull'intervallo $x \in [0, 2\pi]$:

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in [0, 2\pi], \ t \geq 0,$$

con condizioni al contorno

$$u_x(0,t) = u_x(2\pi, t) = 0.$$

Si consideri la condizione iniziale $u(x, t=0) = \sin^2(\frac{x}{2})$ e $\partial_t u(x,t)|_{t=0} = 0$. Si scriva la soluzione a tempi generici $t \geq 0$.

Ex. 5 b)

Le oscillazioni trasversali di una corda ancorata a due punti $x = 0$ e $x = L$ sono descritte dall'equazione delle onde con velocità di propagazione c:

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0, \quad x \in [0, L]. \tag{1.6}$$

Gli estremi della corda sono fissati. Al tempo $t = 0$, la corda viene posta nella condizione iniziale

$$u(x,0) = 0, \quad u_t(x,0) = (L^2 - x^2)^2, \quad x \in [0, L]. \tag{1.7}$$

- (10 punti) Si scriva l'evoluzione temporale della soluzione per $t \geq 0$ usando il metodo di Fourier. I coefficienti di Fourier vanno specificati come integrali espliciti, ma non è necessario calcolare gli integrali.
- (1 punto) Sia T il periodo temporale della soluzione precedente per $t > 0$. Si faccia un esempio di condizione iniziale che porti a oscillazioni di periodo $T/3$.

Ex. 5 c)

Le oscillazioni trasversali di una corda ancorata a due punti $x = 0$ e $x = L$ sono descritte dall'equazione delle onde con velocità di propagazione c:

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0, \quad x \in [0, L]. \tag{1.8}$$

Gli estremi della corda sono fissati. Al tempo $t = 0$, la corda viene posta nella condizione iniziale

$$u(x,0) = 0, \quad u_t(x,0) = (L^2 - x^2)^2, \quad x \in [0, L]. \tag{1.9}$$

- (10 punti) Si scriva l'evoluzione temporale della soluzione per $t \geq 0$ usando il metodo di Fourier. I coefficienti di Fourier vanno specificati come integrali espliciti, ma non è necessario calcolare gli integrali.
- (1 punto) Sia T il periodo temporale della soluzione precedente per $t > 0$. Si faccia un esempio di condizione iniziale che porti a oscillazioni di periodo $T/3$.

Ex. 6 - Vibrations of a drum

Consider a circular drum of radius R. The propagation speed for waves on the drum is v.

- What are the vibration frequencies of the drum?
- Suppose that at time $t = 0$ the displacement of the drum's membrane is $u(r, \theta, t = 0) = (r^2 - R^2)\cos\theta$, and its initial velocity is $\partial_t u(r, \theta, t = 0) = 0$. Write an explicit solution using the eigenfunctions method for the displacement of the membrane. The coefficients of the series can be defined as explicit integrals, but there is no need to compute the integrals or sum the series.

Hint: To recall the form of the eigenfunctions in this case, see notes at the end of the file.

Ex. 7 - A different drum

Consider a second drum (with the same propagation speed v) which has the shape of a quarter of a disk, namely in radial coordinates, $0 \le r \le R$, $0 \le \theta \le \frac{\pi}{2}$.

- What are the vibration frequencies of the drum? How would they change if we had a drum with the shape of a circle sector with angle α?
- Suppose that at time $t = 0$ the displacement of the drum's membrane is $u(r, \theta, t = 0) = 0$, and its initial velocity is $\partial_t u(r, \theta, t = 0) = \sin(2\theta)(r - R) + \sin(4\theta)(r - R)^2$. Write an explicit solution using the eigenfunctions method for the displacement of the membrane.

1.3 Laplace/Poisson equation

Ex. 8 - Dirichlet rectangle

A square metal plate of side π $(0 \le x,\ y \le \pi)$, has its sides kept at the following temperatures:

$$T(x,0) = T(\pi,y) = 0, \quad T(0,y) = \sin y, \quad T(x,\pi) = 5\sin 2x - 7\sin 8x, \quad 0 \le x,\ y \le \pi. \tag{1.10}$$

What is the equilibrium temperature of the plate?

Ex. 9 - Neumann rectangle

Consider the Laplace equation $u_{xx} + u_{yy} = 0$ on a rectangular domain of sides L, M $(0 \le x, \le L$, $0 \le y \le M)$, with Neumann boundary conditions

$$u_x(0,y) = u_x(L,y) = u_y(x,M) = 0, \quad u_y(x,0) = f(x). \tag{1.11}$$

Write a form for the solution using Fourier's method. Specify any consistency conditions that are required.

Ex. 10 - Disk

Consider Laplace equation $\Delta^{(2D)}u = 0$ in the interior of a disk of radius R, with boundary condition $u(R,\theta) = f(\theta)$ on the boundary of the disk.

- Using the Green's function method discussed in the lecture, write down explicitly the solution at an interior point $u(r,\phi)$, as a function of the boundary data. (NOTE: we did not treat the Green's function method for the disk, so ignore this question).
- Derive an alternative form of the solution using the Fourier method.

Ex. 11 - Annulus

Consider Laplace equation in the interior of an annulus region, namely the region defined by $R_1 < r < R_2$, $r = \sqrt{x^2 + y^2}$.

Consider the case $R_2 = 2$, $R_1 = 1$. Using a decomposition into eigenfunctions in radial coordinates, solve the following boundary value problem:*

$$\Delta^{(2D)}u = 0, \quad R_1 < r < R_2, \quad u(R_1,\theta) = \sin 2\theta, \quad u(R_2,\theta) = 0. \tag{1.12}$$

Ex. 12 - Half Disk

Si consideri l'equazione di Laplace,

$$u_{xx} + u_{yy} = 0, \tag{1.13}$$

in un dominio costituito dalla metà di un disco: $x^2 + y^2 \leq R^2$, $x \geq 0$, con condizioni di bordo

$$u(0,y) = 0, \quad y \in [-R, R], \tag{1.14}$$

$$u(R\cos\theta, R\sin\theta) = \cos(3\theta) + \cos\theta, \quad -\frac{\pi}{2} \leq \theta \leq +\frac{\pi}{2}. \tag{1.15}$$

Si determini la soluzione per u all'interno del dominio.

1.4 Relevant formulas

1.4.1 Fourier orthogonality

The following orthogonality properties of the sine/cosine functions are the basis of Fourier analysis:

$$\frac{1}{L}\int_0^{2L} \sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{2L}) = \delta_{mn}, \quad n, m \in \mathbb{N}^+ \tag{1.16}$$

$$\frac{1}{L}\int_0^{2L} \cos(\frac{n\pi x}{L})\cos(\frac{m\pi x}{2L}) = \frac{\delta_{mn}}{2^{\delta_{m,0}}}, \quad n, m \in \mathbb{N}, \tag{1.17}$$

$$\frac{1}{L}\int_0^{2L} \cos(\frac{n\pi x}{L})\sin(\frac{m\pi x}{2L}) = 0. \tag{1.18}$$

*The boundary condition is simplified with respect to the original file, just to make the calculation shorter. One could solve the general case with exactly the same method, i.e. the inner and outer boundary conditions can be any functions.

In some types of applications we often need the following relations:

$$\frac{2}{L}\int_0^L \sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{2L}) = \delta_{mn}, \quad n,m\in\mathbb{N}^+, \tag{1.19}$$

$$\frac{2}{L}\int_0^L \cos(\frac{n\pi x}{L})\cos(\frac{m\pi x}{2L}) = \frac{\delta_{mn}}{2^{\delta_{m,0}}} \quad n,m\in\mathbb{N},, \tag{1.20}$$

which are a simple consequence of the two above (notice that instead (1.18) above does not generalize to the half-interval).

In problems with mixed Dirichlet-Neumann boundary conditions, the following are useful:

$$\frac{2}{L}\int_0^L \sin(\frac{(n+\frac{1}{2})\pi x}{L})\sin(\frac{(n+\frac{1}{2})\pi x}{2L}) = \delta_{mn}, \quad n,m\in\mathbb{N}, \tag{1.21}$$

$$\frac{2}{L}\int_0^L \cos(\frac{(n+\frac{1}{2})\pi x}{L})\cos(\frac{(n+\frac{1}{2})\pi x}{2L}) = \delta_{mn}, \quad n,m\in\mathbb{N}, \tag{1.22}$$

The standard Fourier decomposition of a function defined on an interval $[0,2L]$ involves both sine and cosine functions.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(n\pi x/L) + b_n\sin(n\pi x/L)\right). \tag{1.23}$$

this defines a function of period $2L$. The coefficients can be found explicitly using orthogonality. For instance, integrating $\sin(\frac{n\pi x}{L})$ against the function, and using the orthogonality above, we deduce $b_n = \frac{1}{L}\int_0^{2L} dx\ \sin(\frac{n\pi x}{L})f(x)$.

Often in PDE problems we want to use a different type of series expansion, which is needed to capture the boundary conditions. Typically, we use a basis of trigonometric functions which corresponds to the standard Fourier decomposition of a *larger* interval.

For instance, consider a function $f(x)$ defined on the interval $x\in[0,L]$. In problems where we want to impose Dirichlet-Dirichlet boundary conditions, we use an expansion as a *sine-series*:

$$f(x) = \sum_{n=1}^{\infty} d_n \sin(n\pi x/L). \tag{1.24}$$

Notice that this defines a function of period $2L$, which is odd: $f(x) = -f(-x)$. It is the *odd extension* of the original function. The coefficients are given by $d_n = \frac{2}{L}\int_0^L dx\ \sin(\frac{n\pi x}{L})f(x)$, as can be deduced using the orthogonality relations (1.19) above.

1.4.2 Useful Green's functions

Linear equation 1st order For the ordinary differential equation

$$y'(t) + k^2 y(t) = F(t),$$

with $y(0)=0$, the solution is

$$\int_0^t dse^{-k^2(t-s)}F(s)ds. \tag{1.25}$$

Linear equation 2nd order For the ordinary differential equation

$$y''(t) + \omega^2 y(t) = F(t),$$

with $y(0) = 0$, $y'(0) = 0$, the solution is

$$\frac{1}{\omega}\int_0^t ds \sin(\omega(t-s))F(s)ds. \tag{1.26}$$

1.4.3 Radial coordinates

Radial coordinates in 2D The Laplace operator in $2D$ is $\Delta^{(2D)} \equiv \partial_x^2 + \partial_y^2$. In radial coordinates it becomes:

$$\partial_r^2 + \frac{\partial_r}{r} + \frac{\partial_\theta^2}{r^2}, \tag{1.27}$$

where the radial coordinates are $r = \sqrt{x^2 + y^2}$, $\theta = \arccos\frac{x}{r} = \arcsin\frac{y}{r}$.

Notable radial equation in 2D - Laplace The ODE

$$R''(r) + \frac{R'(r)}{r} - \frac{m^2 R(r)}{r^2} = 0 \tag{1.28}$$

arises studying Laplace equation $\Delta u = 0$ in radial coordinates in 2D.

This equation has two independent solutions: $R(r) = r^{\pm m}$ for $m > 0$, and for $m = 0$ the two independent solutions are: $R(r) = \text{const}$, $R(r) = \log r$.

Notable radial equation in 2D - Helmoltz The ODE

$$R''(r) + \frac{R'(r)}{r} + \frac{\lambda\, r^2\, R(r) - m^2 R(r)}{r^2} = 0 \tag{1.29}$$

arises studying Helmoltz equation $\Delta u = -\lambda u$ in radial coordinates in 2D.

It can be transformed into Bessel equation for $y(x) = R(\sqrt{\lambda}x)$: $y''(x) + \frac{y'(x)}{x} + \frac{x^2 - m^2}{x^2} y(x) = 0$.

Assume $\mathrm{Re}(m) \geq 0$. Then the solution of Bessel equation with behaviour x^m for $x \sim 1$ is called *Bessel function of the first kind* $J_m(x)$. It has infinitely many zeros on the positive real axis (including $x = 0$ if $m > 0$).

Bessel functions satisfy the following orthogonality relations:

$$\int_0^1 x J_\alpha(\mu_{\alpha,k} x) J_\alpha(\mu_{\alpha,l} x) \propto \delta_{kl}, \tag{1.30}$$

$\forall \alpha$, $k = 1, 2, \ldots$, where $\mu_{\alpha,k}$ denote the zeros, $J_\alpha(\mu_{\alpha,k}) = 0$, $k = 1, 2, \ldots$.

Radial coordinates in 3D The Laplace operator in $3D$ is $\Delta^{(3D)} \equiv \partial_x^2 + \partial_y^2 + \partial_z^2$. In radial coordinates it becomes:

$$\partial_r^2 + \frac{2\partial_r}{r} + \frac{\partial_\theta^2 + \cot\theta\partial_\theta + \frac{\partial_\phi^2}{\sin^2\theta}}{r^2}, \tag{1.31}$$

where the radial coordinates are $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \arccos\frac{z}{r}$, $\phi \arcsin\frac{y}{r\sin\theta} = \arccos\frac{x}{r\sin\theta}$.

Notable radial equation in 3D - Laplace The ODE

$$R''(r) + \frac{2R'(r)}{r} - \frac{l(l+1)R(r)}{r^2} = 0, \tag{1.32}$$

arises studying Laplace equation $\Delta u = 0$ in radial coordinates in 3D.

This equation has two independent solutions: $R(r) = r^l$ and r^{-l-1}.

Notable radial equation in 3D - Helmoltz Studying the more general Helmoltz equation, $\Delta^{(3D)} u = -\lambda u$, the radial part leads to the ODE

$$R''(r) + \frac{2R'(r)}{r} - \frac{l(l+1)R(r) - \lambda r^2 R(r)}{r^2} = 0, \tag{1.33}$$

This equation can also be mapped to Bessel equation doing the substitution $y(x) = R(\sqrt{\lambda}x)/\sqrt{x}$. **check it!**

Assume $\mathrm{Re}(l) \geq 0$. The solution with behaviour $R(r) \sim r^l$ can be written as $R(r) \propto \frac{J_{l+\frac{1}{2}}(\sqrt{\lambda}r)}{\sqrt{\lambda\, r}}$.

Note: The simplest case is for $l = 0$ (which is relevant to decompose the angle-independent part of solutions of Helmoltz's equation on the sphere). In this case, the eigenfunction is simple because $J_{\frac{1}{2}}(x)/\sqrt{x} = \sqrt{\frac{2}{\pi}} \frac{\sin x}{x}$.