ESERCIZI : ODE, PROB. VALORI INIZIALI

1

$$y'' = -\omega^2 y + t^3 \qquad y(0) = 3$$

$$y'(0) = 5$$

$y_1(t) = \sin \omega t$

$y_2(t) = \cos \omega t$

$$W = \begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix}$$

$$|W| = -\omega \neq 0$$

$$\begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 & 1/\omega \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5/\omega \\ 3 \end{pmatrix}$$

GREEN'S FUNCTION:

for $t > 0$

$$G(t,\tau) = + \frac{\begin{vmatrix} \sin \omega t & \cos \omega t \\ \sin \omega \tau & \cos \omega \tau \end{vmatrix}}{\omega} \Theta(t-\tau) \cdot \Theta(\tau)$$

$$= \frac{1}{\omega} \sin \omega (t-\tau) \, \Theta(t-\tau) \, \Theta(\tau)$$

SOLUZIONE

$$y(t) = \frac{1}{\omega} \int_0^t \sin[\omega(t-\tau)]\,\tau^3\,d\tau + \frac{5}{\omega}\sin\omega t + 3\cos\omega t$$

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(2)
$$y'' - 2y' + y + \underline{\sin(t)} = 0$$

$-\sin(t)$: TERMINE INOMOGENEO

$$y(0) = 0, \quad y'(0) = 0$$

BASE SOLUZIONI:
$$\begin{cases} y_1(t) = e^t \\ y_2(t) = t\,e^t \end{cases}$$

$$\rho^2 - 2\rho + 1 = 0 \qquad \rho = 1$$

$$W(t) = \begin{vmatrix} e^t & t\,e^t \\ e^t & e^t(1+t) \end{vmatrix} = e^{2t}$$

$$G(t,\tau) = -\frac{\begin{vmatrix} e^t & t\,e^t \\ e^\tau & \tau\,e^\tau \end{vmatrix}}{W(\tau)} = -e^{\tau-t}(\tau-t)\,\Theta(t-\tau)\cdot\Theta(\tau)$$

GREEN'S FUNCTION

(for $t>0$)

$$y(t) = -\int_0^t e^{\tau-t}(t-\tau)\,\underline{\sin(\tau)}\,d\tau$$

3)

$$y'(t) - t^2\, y(t) = \underline{\underline{\cos^2(t)}}$$

TERMINE INOMOG.

SOL. EQ. OMOGENEA: $\longrightarrow \dfrac{dy_1}{y_1} = +t^2$

$$y_1(t) \equiv e^{t^3/3}$$

F. DI GREEN (PER $t > 0$):

$$G(t, \tau) = \frac{y_1(t)}{y_1(\tau)}\, \Theta(t-\tau)\, \Theta(\tau)$$

$$\rightarrow \boxed{y(t) = \int_0^t d\tau\, e^{\frac{t^3-\tau^3}{3}} \cdot \underline{\underline{\cos^2(\tau)}}\, d\tau}$$

PROBLEMI A VALORI DI BORDO

4. $y'' = -4y$ $\qquad y(0) = A$, $\quad y'(\pi) = B$

$y_1(t) = \sin 2t$

$y_2(t) = \cos 2t$

MATRICE COND. DI BORDO

$$M = \begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(\pi) & y_2'(\pi) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ +2 & 0 \end{pmatrix}$$

$|M| \neq 0 \to \exists!$ SOLUZIONE $\forall (A, B)$.

SOL. $y(t) = C_1 y_1(t) + C_2 y_2(t)$ CON:

$$M \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \to \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 & +\frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} +\frac{B}{2} \\ A \end{pmatrix}$$

$$\to y(t) = +\frac{B}{2} \sin(2t) + A \cos(2t).$$

5|

$$y''(t) = -4\, y(t) \qquad y(0) = A$$

$$y(\pi) + 3\, y'(\pi) = B$$

$$y_1(t) = \sin(2t) \qquad y_2(t) = \cos(2t) \longrightarrow M = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$$

$$\boxed{|M| \neq 0 \to \exists! \text{ SOLUZIONE}}$$

$$\to y(t) = C_1\, y_1(t) + C_2\, y_2(t)$$

CON

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = M^{-1} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} & \frac{1}{6} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$= \begin{pmatrix} \frac{B-A}{6} \\ A \end{pmatrix}$$

$$y(t) = \frac{(B-A)}{6} \sin 2t + A \cos 2t$$

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6 |

$$y'' + y = \underline{1+t}$$ TERMINE INOMOG.

$$\begin{cases} y(0) = A \\ y\left(\frac{3}{2}\pi\right) = B \end{cases}$$

$y_1(t) = \sin t$

$y_2(t) = \cos t$

$$M = \begin{pmatrix} y_1(0) & y_2(0) \\ y_1\left(\frac{3}{2}\pi\right) & y_2\left(\frac{3}{2}\pi\right) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$|M| \neq 0 \rightarrow \exists!$ SOLUZIONE!

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + \int_0^{\frac{3}{2}\pi} G(t,\tau)\,\underline{(1+\tau)}\,d\tau$$

F. DI GREEN.

DOVE

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = M^{-1} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -B \\ A \end{pmatrix},$$

PER TROVARE F. DI GREEN LA SCRIVIAMO COME:

$$G(t,\tau) =$$

$$\Theta(t-\tau)\cdot\left[D_1 \sin t + D_2 \cos t \right]$$

$$+ \Theta(\tau-t) \left[\tilde{D}_1 \sin t + \tilde{D}_2 \cos t \right]$$

IMPONENDO COND. DI BORDO: $G(t,\tau)\big|_{t=0} = G\big|_{t=\frac{3}{2}\pi} = 0$

$$D_2 = 0 \quad , \quad \tilde{D}_1 = 0$$

$$\rightarrow G = \Theta(t>\tau)\cdot \tilde{D}_2 \cos t + \Theta(\tau>t) D_1 \sin t$$

IMPONENDO CONTINUITÀ A $t=\tau$:

$$\tilde{D}_2 \cos\tau = D_1 \sin\tau \quad \rightarrow \tilde{D}_2 = D_1 \tan\tau$$

IMPONENDO SALTO IN $\frac{\partial}{\partial t} G \Big|_{t=\tau} = 1$:

$$-\tilde{D}_2 \sin\tau - D_1 \cos\tau = 1$$

$$D_1 \cdot (-\cos^2\tau - \sin^2\tau) = \cos\tau \rightarrow D_1 = -\cos\tau$$

$$\to \boxed{\begin{aligned} G &= -\Theta(t>\tau)\, \sin\tau \cos t \\ &\quad - \Theta(\tau > t)\, \sin t \cos\tau \end{aligned}}$$

$$\to$$

$$y(t) = C_1 \sin t + C_2 \cos t$$

$$- \int_0^t d\tau \, \cos t \sin\tau \, (1+\tau)\, d\tau$$

$$- \int_t^{\frac{3}{2}\pi} d\tau \, \cos\tau \sin t \, (1+\tau)\, d\tau \, .$$

7 | WE CAN VERIFY:

- CONDIZIONI DI BORDO:

$$G(x,t)\Big|_{x=0} = G(x,t)\Big|_{x=1} = 0$$

- CONTINUITA':

$$\lim_{x\to t^-} G(x,t) = \lim_{x\to t^+} G(x,t)$$

- SALTO DI +1 NELLA DER. PRIMA:

$$\lim_{x\to t^+} \partial_x G(x,t) = 1 + \lim_{x\to t^-} \partial_x G(x,t)$$

LA SOL. DEL PROBLEMA RICHIESTO:

$$y''(x) - y(x) = 2\sin(x)$$

$$y(0) = 0 \qquad y(1) = 2 \to M = \begin{pmatrix} y_1(0) & y_2(0) \\ y_1(1) & y_2(1) \end{pmatrix}$$

È DATA DA:

$$= \begin{pmatrix} 1 & 1 \\ e & e^{-1} \end{pmatrix}$$

$$y(x) = C_1 e^{x} + C_2 e^{-x}$$

$$+ 2\int_0^x d\tau \, \frac{\sinh(x)\sinh(\tau-1)}{\sinh(1)} \sin\tau$$

$$+ 2\int_x^1 d\tau \, \frac{\sinh(\tau)\sinh(x-1)}{\sinh(1)} \sin\tau ;$$

CON

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = M^{-1} \cdot \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -e^{-1} & 1 \\ +e & -1 \end{pmatrix} \frac{1}{\sinh(1)} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$= \frac{1}{\sinh(1)} \cdot \begin{pmatrix} 2 \\ -2 \end{pmatrix} .$$

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8 | IMPONENDO LE CONDIZIONI APPROPRIATE, SI TROVA:

$$K_2 = +1 \qquad \to G(x,t) = -\frac{1}{2} e^{-|x-t|}$$

$$K_1 = \frac{1}{2}$$

INVECE PER IL PROBLEMA A VAL. INIZIALI CON CONDIZIONI

$$y(0) = 0, \quad y'(0) = 0$$

LA F. DI GREEN SAREBBE:

$$G(x,t) = -\frac{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1(t) & y_2(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}}$$

CON BASE $y_1 = e^{x}$, $y_2 = e^{-x}$ $\rightarrow W = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -2$

$$\rightarrow G(x,t) = \frac{1}{2} \cdot \begin{vmatrix} e^{x} & e^{-x} \\ e^{t} & e^{-t} \end{vmatrix} = \sinh(x-t)$$

LA SOL. DEL PROBLEMA

$$y'' - y = \sin t \quad \text{CON}$$

$$y(0) = 1 \;,\; y'(0) = 0 \qquad \text{E'}$$

$$y(t) = \cosh t + \int_0^t d\tau \, \sinh(t-\tau) \sin\tau .$$

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9) PER RISOLVERE

$$y'' = e^{t^2} \qquad \begin{cases} y(0) = 0 \\ y(1) = 1 \end{cases}$$

NOTIAMO CHE LE SOL. EQ.

OMOGENEA SONO GENERATE DA:

$$y_1(t) = 1 \qquad y_2(t) = t .$$

MATRICE COND. DI BORDO :

$$M = \begin{pmatrix} y_1(0) & y_2(0) \\ y_1(1) & y_2(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$|M| \neq 0$ ✓

$$M^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

F. DI GREEN :

$$G(t,\tau) = A\,t\ \Theta(\tau > t) + B\,(t-1)\ \Theta(t-\tau)$$

COND. DI BORDO ✓

CONTINUITA' : $A\tau = B(\tau - 1) \rightarrow B = A\dfrac{\tau}{\tau - 1}$.

SALTO DERIV. PRIMA:

$$B - A = 1$$

$$\rightarrow \quad B = 1 + A$$

$$A\,\tau = (1 + A)(\tau - 1)$$

$$\tau - 1 - A = 0 \quad \rightarrow \begin{cases} A = \tau - 1 \\ B = \tau \end{cases}$$

$$\rightarrow G(t,\tau) = t \cdot (\tau - 1) \cdot \Theta(\tau > t) + \tau\,(t - 1)\,\Theta(t > \tau)$$

$\rightarrow$ SOLUZIONE:

$$y(t) = C_1 + C_2\, t$$

$$+ \int_0^t d\tau \; e^{\tau^2} (t-1)\,\tau$$

$$+ \int_t^1 d\tau \; e^{\tau^2} (\tau-1)\, t$$

DOVE

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \underline{1} \end{pmatrix}.$$