ORDINARY DIFFERENTIAL EQUATIONS (O.D.E.)

AN ORDINARY DIFFERENTIAL EQUATION (ODE) OF ORDER n IS A RELATION OF THE FORM:

$$F\left(\frac{d^n}{dx^n}y(x), \frac{d^{n-1}}{dx^{n-1}}y(x), \dots, \frac{dy(x)}{dx}, y(x), x\right) = 0$$

(IMPLICIT FORM)

OR:

$$\frac{d^n y(x)}{dx^n} = G\left(\frac{d^{n-1}}{dx^{n-1}}y(x), \dots, \frac{dy(x)}{dx}, y(x), x\right)$$

(EXPLICIT FORM)

THE EQUATION IS CALLED "AUTONOMOUS" IF THERE IS NO EXPLICIT DEPENDENCE ON x,

FOR EXAMPLE $y'' + \sin(y'y) = 0$ IS AUTONOMOUS.

IN MANY (BUT NOT ALL) APPLICATIONS, x HAS THE INTERPRETATION OF TIME.

TYPICALLY WE ARE INTERESTED IN SOLVING AN INITIAL VALUE PROBLEM (IVP):

FIND A SOLUTION OF THE ODE

$$0 = F\left(\frac{d^n}{dx^n} y, \frac{d^{n-1}}{dx^{n-1}} y, \dots, y', y, x\right) = 0$$

SUCH THAT

$$\begin{cases} y(x_0) = Y_0 \\ y'(x_0) = Y_1 \\ \vdots \\ \dfrac{d^{n-1} y}{dx^{n-1}}(x_0) = Y_{n-1} \end{cases}$$

SOLUTIONS IN GENERAL COME IN FAMILIES, DEPENDING ON n CONTINUOUS PARAMETERS. THE PARAMETERS CAN BE FIXED IN TERMS OF THE INITIAL CONDITIONS $Y_0, Y_1, \dots, Y_{n-1}$.

REWRITING n-th ORDER EQUATION AS A SYSTEM OF 1st ORDER ODE'S

START FROM EXPLICIT FORM:

$$\frac{d^n}{dx^n} y(x) = G\left(\frac{d^{n-1}y}{dx^{n-1}}, \dots, \frac{dy}{dx}, y, x\right)$$

DEFINE A VECTOR FUNCTION:

$$\vec{F}(x) = \begin{pmatrix} F_0(x) \\ F_1(x) \\ \vdots \\ F_{n-1}(x) \end{pmatrix} \equiv \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ \frac{d^{n-1}}{dx^{n-1}} y(x) \end{pmatrix}$$

THEN IT SATISFIES A 1st ORDER ODE.

$$\frac{d}{dx}\vec{F}(x) = \begin{pmatrix} F_0'(x) \\ F_1'(x) \\ \vdots \\ F_{n-2}'(x) \\ F_{n-1}'(x) \end{pmatrix} = \begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_{n-1}(x) \\ G(F_{n-1}, F_{n-2}, \dots, F_0, x) \end{pmatrix}$$

LOCAL EXISTENCE & UNIQUENESS OF SOLUTIONS

$$\frac{d^n}{dx^n} y(x) = G\left(y^{(n-1)}, \dots, y', y, x\right)$$

CONTINUOUS WITH CONTINUOUS DERIVATIVES IS MORE THAN ENOUGH

IF G IS "SMOOTH ENOUGH" IN A CLOSED NEIGHBOURHOOD OF A POINT

$$x = x_0, \quad y = Y_0, \quad y' = Y_1, \quad \dots, \quad y^{(n-1)} = Y_{n-1},$$

THEN THE INITIAL VALUE PROBLEM WITH INITIAL CONDITION

$$y(x_0) = Y_0, \quad y'(x_0) = Y_1, \quad \dots, \quad y^{(n-1)}(x_0) = Y_{n-1}$$

HAS A UNIQUE SOLUTION IN SOME INTERVAL $[x_0 - \varepsilon, x_0 + \varepsilon]$, $\varepsilon > 0$.

(BUT WE DON'T KNOW A PRIORI HOW LARGE ε CAN GET).

LINEAR ODE's

$$\frac{d^n y(x)}{dx^n} a_n(x) + \frac{d^{n-1}}{dx^{n-1}} y(x)\, a_{n-1}(x) + \cdots$$

$$+ \cdots \frac{dy(x)}{dx} a_1(x) + y(x)\, a_0(x) = 0$$

(A LINEAR COMBINATION OF THE FUNCTIONS AND ITS DERIVATIVES, WITH COEFFICIENTS WHICH CAN BE FUNCTIONS OF x, <u>BUT NOT OF y</u>)

THIS IS CALLED A <u>LINEAR</u> <u>HOMOGENEOUS</u> ODE.

IT IS SIMPLE TO PROVE THAT FOR SUCH EQUATIONS ANY LINEAR COMBINATION OF SOLUTIONS IS STILL A SOLUTION, i.e. IF $y_1(x)$, $y_2(x)$ SOLVE THE ODE, THEN $C_1 \, y_1(x) + C_2 \, y_2(x)$ IS ALSO A SOLUTION (WITH <u>CONSTANT</u> C_1, C_2).

THE GENERAL SOLUTION IS OBTAINED BY TAKING A LINEAR COMBINATION OF n INDEPENDENT SOLUTIONS.

IF WE HAVE n SOLUTIONS $y_1(x), \dots, y_n(x)$, WE CAN CHECK THEIR LINEAR INDEPENDENCE BY COMPUTING THE WRONSKIAN:

$$W = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & & y_n^{(n-1)}(x) \end{vmatrix} \longleftarrow \text{DETERMINANT}$$

THE SOLUTIONS ARE INDEPENDENT IF AND ONLY IF $W \neq 0$.

IN GENERAL, W IS A FUNCTION OF x BUT IT SATISFIES A SIMPLE 1st ORDER ODE. WE WILL SEE IT EXPLICITLY IN THE $n=2$ CASE.

GENERAL EQUATION FOR $W(x)$

IT IS POSSIBLE TO PROVE IN GENERAL:

$$\frac{dW(x)}{dx} = -\frac{a_{n-1}(x)}{a_n(x)} W(x)$$

FROM WHICH ONE CAN SOLVE:

$$W(x) = W(x_0)\, e^{-\int_{x_0}^{x} \frac{a_{n-1}(x')}{a_n(x')} dx'}$$

(BELOW WE SEE IT MORE EXPLICITLY FOR $n=2$).

FOR ANY REFERENCE POINT x_0.

IF $W(x)$ IS $\neq 0$ AT ANY POINT, THEN IT IS $\neq 0$ EVERYWHERE.

INHOMOGENEOUS CASE

$$\left(\frac{d^n}{dx^n} y(x)\right) a_n(x) + \frac{d^{n-1}}{dx^{n-1}} y(x)\, a_{n-1}(x) + \cdots$$
$$+ \cdots + a_0(x)\, y(x) = r(x)$$

FOR $r(x) \neq 0$, WE CAN FIND THE GENERAL SOLUTION BY:

- STUDYING THE ASSOCIATED HOMOGENEOUS EQUATION (WITH $r(x) \to 0$) AND FINDING n INDEPENDENT SOLUTIONS $y_1(x), \ldots, y_n(x)$.

- FINDING A PARTICULAR SOLUTION $y_{inh}(x)$ OF THE INHOMOGENEOUS EQUATION.

THEN THE GENERAL SOLUTION HAS THE FORM:

$$C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) + y_{inh}(x)$$

(i.e., GENERAL SOLUTION OF HOMO-
GENEOUS EQUATION + PARTICULAR
SOLUTION OF INHOMOGENEOUS).

SIMPLE ANALYTIC METHODS TO SOLVE ODE'S EXPLICITLY

* 1st ORDER SEPARABLE EQUATION

$$y'(x)\, A(x)\, B(y) + C(x)\, D(y) = 0$$

$$\hookrightarrow \quad y' \frac{B(y)}{D(y)} = - \frac{C(x)}{D(x)}$$

$$\hookrightarrow \quad \int^{y} \frac{B(s)}{D(s)}\, ds = - \int^{x} \frac{C(t)}{D(t)}\, dt + K$$

constant

* IN PARTICULAR, OF COURSE

$$y' = A(x)$$

$$\hookrightarrow \quad y(x) = \int^{x} A(t)\, dt + K$$

* EQUATION IN THE FORM OF EXACT DIFFERENTIAL

$$y' A(x,y) + B(x,y) = 0$$

WHERE $A(x,y) = \partial_y V(x,y)$

$B(x,y) = \partial_x V(x,y)$

FOR SOME POTENTIAL FUNCTION $V(x,y)$.

IF THIS IS TRUE, THEN THE ODE TELLS US THAT

$$y' \partial_y V + \partial_x V = 0$$

$$\longrightarrow \boxed{\frac{d}{dx} V(x, y(x)) = 0 \quad \text{ON SOLUTIONS OF THE ODE}}$$

THIS MEANS THAT SOLUTIONS LIVE ON LEVEL CURVES OF V.

$\downarrow$

$K = V(x, y(x))$ ← IMPLICIT FORM OF SOLUTION

HOW TO USE THE METHOD? IF WE HAVE $A(x,y)$, $B(x,y)$, WE CAN CHECK IF THEY SATISFY

$$\boxed{\partial_x A(x,y) = \partial_y B(x,y)}$$

NECESSARY CONDITION FOR V TO EXIST,

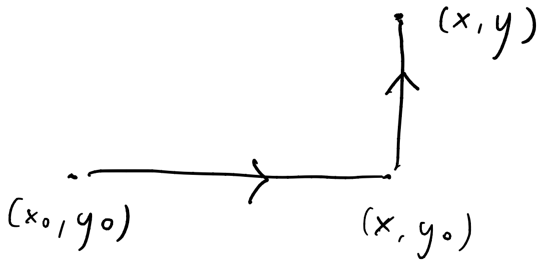
BECAUSE
$$\partial_x A = \partial_x \partial_y V$$
$$\partial_y B = \partial_y \partial_x V$$

· MUST BE EQUAL ASSUMING C^2 FUNCTIONS

IF THE NECESSARY CONDITION IS SATISFIED, (IN A SIMPLY-CONNECTED REGION) WE CAN RECONSTRUCT $V(x,y)$ BY INTEGRATING ITS GRADIENT FIELD

$$(\partial_x V, \partial_y V) = (B(x,y), A(x,y))$$

WE CAN TAKE ANY INITIAL POINT (x_0, y_0), AND INTEGRATING, FOR EXAMPLE ALONG THE CONTOUR:

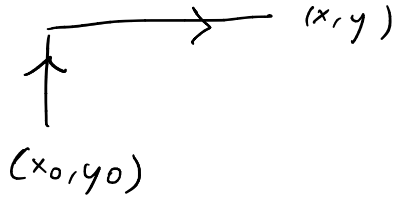


WE FIND $V(x, y) - V(x_0, y_0)$

$$= \int_{x_0}^{x} \partial_x V(s, y_0)\, ds + \int_{y_0}^{y} dt\, \partial_y V(x, t)$$

$$= \int_{x_0}^{x} B(s, y_0)\, ds + \int_{y_0}^{y} dt\, A(x, t).$$

EXERCISE: VERIFY THAT CHOOSING THE CONTOUR



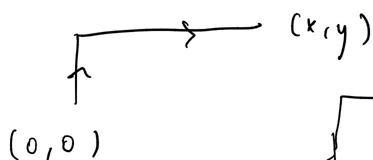
ONE FINDS THE SAME RESULT.

EXAMPLE

ODE: $\underbrace{(3x^2-y)}_{B} + \underbrace{(2y-x)}_{A}\,y' = 0$

TEST: $\partial_y B = \partial_x A = -1$ ✓

THEN WE BUILD:



$$\boxed{\begin{aligned} V(x,y) &= \int_0^y ds\,(2s) + \int_0^x (3t^2-y)\,dt \\ &= y^2 + x^3 - yx \end{aligned}}$$

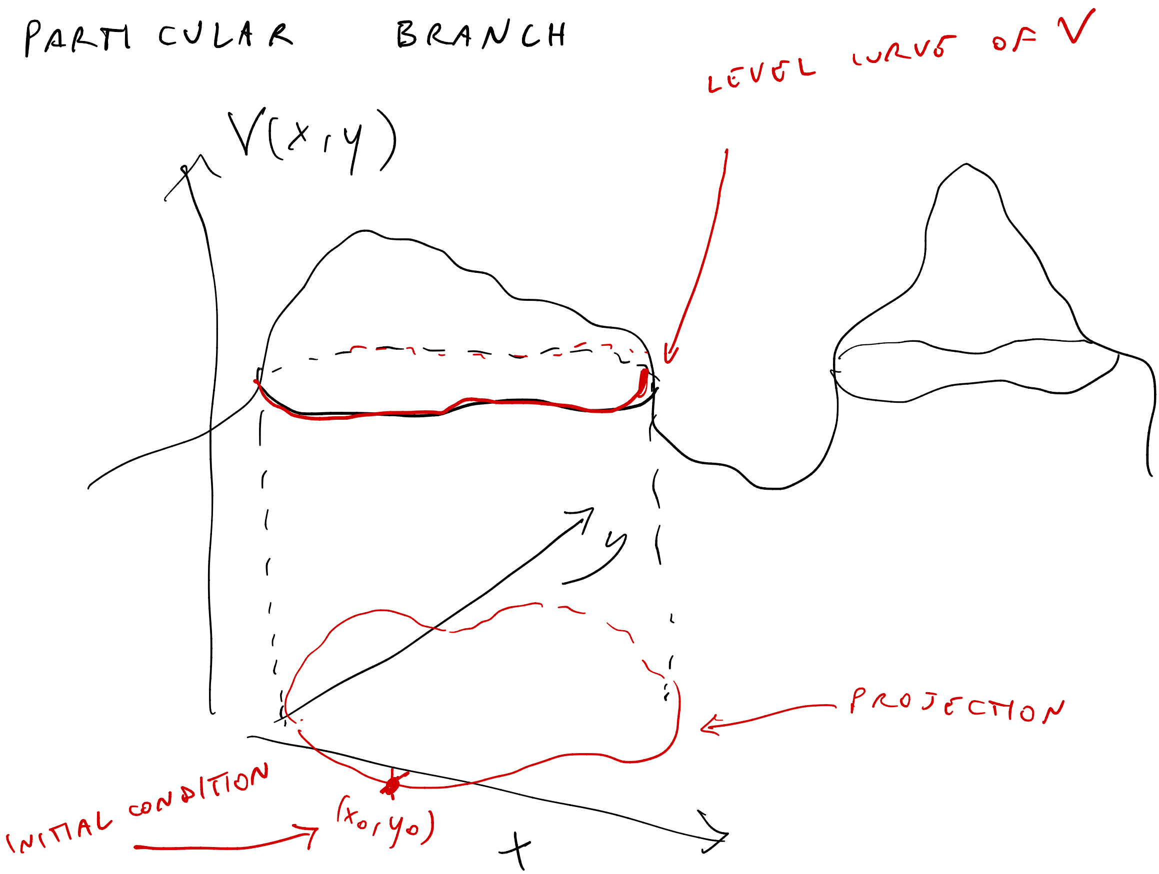
WITHOUT LOSS OF GENERALITY, WE CHOSE $V(0,0)=0$.

(WE CAN ALWAYS REDEFINE V BY A CONSTANT)

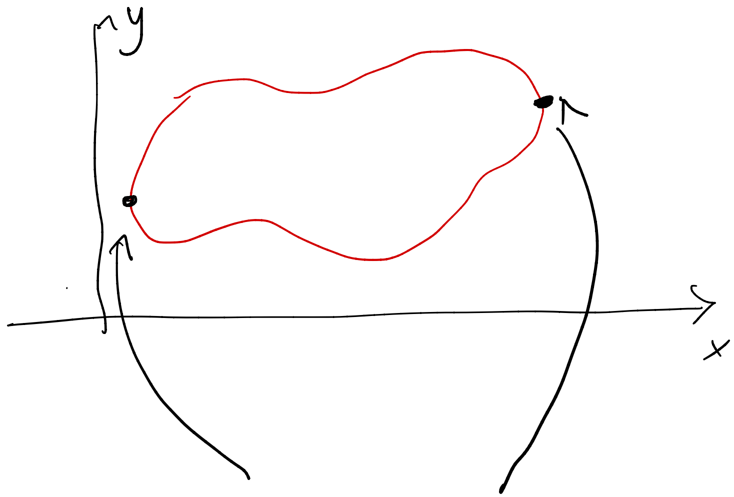
THEN THE SOLUTION OF THE ODE IS GIVEN IMPLICITLY BY:

$$K = y^2 + x^3 + yx$$

NOTICE THAT INVERTING y AS A FUNCTION OF x, WE CAN FIND MORE BRANCHES. IN A CONCRETE I.V.P. WE ARE ON A PARTICULAR BRANCH



THE PROJECTION OF LEVEL CURVES ON THE (x, y) PLANE GIVES THE SOLUTION.



THERE MAY BE SINGULAR POINTS WHERE $y'(x) \to \pm\infty$.

THESE OCCUR FOR VALUES OF (x, y) WHERE THE ASSUMPTIONS OF THE EXISTENCE & UNIQUENESS THEOREM ARE VIOLATED.

IN THE EXAMPLE ABOVE $(3x^2 - y) + (2y - x)y' = 0$

SUCH POINTS ARE ALL POINTS WHERE

$$2y - x = 0$$

THESE ARE THE POINTS WHERE THE COEFFICIENT

IN FRONT OF y' VANISHES.

POINTS WHERE THE ORDER OF AN ODE IS REDUCED ARE ALWAYS POTENTIALLY SINGULAR.

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* LINEAR EQ. FIRST ORDER

A SIMPLE ODE LIKE $y' + y + x = 0$ IS NOT SOLVABLE WITH THE METHODS ABOVE.

(CONVINCE YOURSELF THIS IS TRUE!)

WE WILL SOLVE IT WITH THE METHOD OF VARIATION OF CONSTANTS.

LET US CONSIDER IN GENERAL:

$$y' + p(x)\,y = r(x)$$

FIRST, IT IS EASY TO SOLVE THE HOMOGENEOUS CASE WHEN $r = 0$

$$y' + p(x)\, y = 0.$$

THIS IS SEPARABLE AND GIVES:

$$y(x) = C \cdot e^{-\int^{x} p(t)\, dt} \equiv C \cdot y_1(x)$$

TO SOLVE $y' + p(x)\, y = r(x)$, WE LOOK FOR A SOLUTION IN THE FORM:

$$y_{inh}(x) = C(x) \cdot \underbrace{y_1}_{\uparrow}(x)$$

HOMOGENEOUS SOLUTION

$$y_1' + p(x)\, y_1 = 0$$

THEN THE ODE IMPLIES:

$$C' y_1 = r \longrightarrow C(x) = \int_{x_0}^{x} \frac{ds \; r(s)}{y_1(s)}$$

$$\hookrightarrow \quad y_{inh}(x) = \int_{x_0}^{x} ds \; r(s) \, \frac{y_1(x)}{y_1(s)}$$

$$= \int_{x_0}^{x} ds \; r(s) \; e^{-\int_s^x dt \; p(t)}$$

$\rightarrow$ GEN. SOLUTION =

$$A \cdot y_1(x) + y_{inh}(x)$$

$$= A \; e^{-\int_{x_0}^{x} p(t) \, dt} + \int_{x_0}^{x} ds \; r(s) \; e^{-\int_s^x p(t) \, dt} .$$

THE METHOD OF VARIATION OF CONSTANTS IS VERY GENERAL. FOR ALL LINEAR ODE'S, IT ALLOWS TO FIND A SOL. OF INHOMOG. EQUATION STARTING FROM BASIS OF SOLUTIONS OF HOMOG. CASE.

2nd ORDER LINEAR

* HOMOGENEOUS, WITH CONSTANT COEFFICIENTS.

$$y'' + a y' + b = 0$$

↳ LOOK FOR SOLUTIONS IN FORM $y(x) \propto e^{\rho x}$:

↳ $(\rho^2 + a\rho + b) = 0$

CHARACTERISTIC EQUATION

IN GENERAL
2 SOLUTIONS → 2 LIN. INDEP. SOLUTIONS

$$C_1 e^{\rho_+ x} + C_2 e^{\rho_- x}$$

- IF THE EQUATION HAS COINCIDENT ROOTS, i.e. $\rho_+ = \rho_- = \rho$, THEN WE CAN TAKE $e^{\rho x}$ AND $x \cdot e^{\rho x}$ AS BASIS OF SOLUTIONS.

* GENERAL CASE (HOMOGENEOUS)

$$y'' + p(x)\, y' + q(x)\, y = 0$$

- PROPERTIES OF THE WRONSKIAN

FOR TWO SOLUTIONS $y_1(x), y_2(x)$, THE WRONSKIAN IS:

$$W(x) \equiv \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x)$$

- IT IS EASY TO SEE THAT IT SATISFIES:

$$\boxed{W'(x) = -p(x)\, W(x)}$$

$$\hookrightarrow \; W(x) = W(x_0)\, e^{-\int_{x_0}^{x} p(t)\, dt}$$

• RECONSTRUCTING THE SECOND SOLUTION

SUPPOSE WE HAVE (SOMEHOW) FOUND ONE SOLUTION $y_1(x)$.

LET US LOOK FOR $y_2(x)$ IN THE FORM

$$y_2(x) = u(x)\, y_1(x)$$

(VARIATION OF CONSTANTS AGAIN)

THE ODE $y'' + p(x) y' + q(x) y = 0$ SATISFIED BY y_2 IMPLIES:

$$\boxed{u'' y_1 + 2 u' y_1' + p(x) u' y_1 = 0}$$

(WHERE WE DROPPED SOME TERMS BECAUSE y_1 SOLVES THE ODE)

↳ THEN WE CAN DEFINE $V(x) = u'(x)$

↓

$$v' y_1 + v \cdot (2 y_1' + p y_1) = 0$$

(SEPARABLE EQUATION FOR $v(x)$)

$\downarrow$

$$v(x) = C \cdot \frac{e^{-\int^x p(t)\,dt}}{y_1^2(x)} = \tilde{C} \frac{W(x)}{y_1^2(x)}$$

$$v = u' \longrightarrow u(x) = C \int^x \frac{e^{-\int_{x_0}^t p(s)\,ds}}{y_1^2(t)}\,dt$$

$$\hookrightarrow y_2(x) = C \int^x y_1(x) \frac{e^{-\int_{x_0}^t p(s)\,ds}}{y_1^2(t)}\,dt$$

2nd ORDER INHOMOGENEOUS: VARIATION OF CONSTANTS

SUPPOSE WE HAVE TWO SOLUTION y_1, y_2 OF THE HOMOG. EQ:

$$y'' + p(x)\, y' + q(x)\, y = 0,$$

WITH WRONSKIAN $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0.$

WE CAN USE THE VARIATION OF CONSTANTS METHOD TO CONSTRUCT THE SOLUTION OF

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

IDEA: LOOK FOR SOLUTION OF THE FORM:

$$\boxed{y_{inh}(x) = C_1(x)\, y_1(x) + C_2(x)\, y_2(x)}$$

↳ AGAIN, WE TOOK THE GENERAL SOLUTION

$C_1\, y_1(x) + C_2\, y_2(x)$ FOR $r(x)=0$,

AND WE TURNED CONSTANTS INTO FUNCTIONS.

WE WILL ALSO IMPOSE AN EXTRA CONDITION:

$$\boxed{C_1'(x)\, y_1(x) + C_2'(x)\, y_2(x) = 0} .$$

PLUGGING y_{inh} INTO THE ODE, AND USING THE CONSTRAINT ABOVE (AND ITS DERIVATIVE), WE FIND:

$$\boxed{C_1'\, y_1' + C_2'\, y_2' = r(x)}$$

PUTTING TOGETHER THE LAST EQUATION AND THE CONSTRAINT WE FIND:

$$\begin{cases} C_1' \, y_1 + C_2' \, y_2 = 0 \\ C_1' \, y_1' + C_2' \, y_2' = r(x) \end{cases}$$

$\longrightarrow$ SOLVE THE LINEAR SYSTEM FOR C_1', C_2', THEN INTEGRATE TO FIND $C_1(x)$, $C_2(x)$.

$\downarrow$

THE END RESULT IS :

$$y_{inh}(x) = C_1(x)\, y_1(x) + C_2(x)\, y_2(x)$$

$$= \int_{x_0}^{x} H(x,t)\, r(t)\, dt$$

WHERE
$$H(x,t) = \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} .$$

NOTE :

ABOVE, x_0 CAN BE TAKEN ARBITRARILY. CHANGING x_0 JUST ADDS A SOLUTION OF THE HOMOGENEOUS EQUATION TO y_{inh}.

SOME EXERCISES:

* $y' + y + x = 0$ WITH $y(0) = 1$.

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* $y'' + y = 1$ WITH $y(0) = 1$, $y'(0) = 0$.

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* $2xy + y'(y^2 + x^2) = 0$ WITH $y(0) = 1$

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* $y'' + \delta x y' + \delta y = 0$ δ = PARAMETER

  - VERIFY THAT $y_1(x) = e^{-\frac{\delta x^2}{2}}$ IS A SOLUTION
  - FIND A SECOND SOLUTION
  - SOLVE THE I.V.P. WITH $y(0) = 1$, $y'(0) = 3\delta$.