PARTIAL DIFFERENTIAL EQUATIONS (P.D.E.)

A PDE IS AN EQUATION OF THE FORM

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}, u_{x_1 x_1}, \dots, u_{x_i x_j}, \dots) = 0$$

WHERE $u_{x_i x_j \dots x_n} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \dots \frac{\partial}{\partial x_n} u$ etc...

THE ORDER OF THE PDE IS THE ORDER OF THE HIGHEST DERIVATIVE APPEARING.

EXAMPLES OF PDE'S RELEVANT IN PHYSICS:

- $u_x + u_y = 0$ (TRANSPORT EQUATION)
- $u_x + u u_y = 0$ OR $u_x + f(u) u_y = 0$ (NONLINEAR TRANSPORT EQ.)

  EXHIBIT THE PHENOMENON OF "SHOCK WAVES"
- $u_{tt} - u_{xx} = 0$ (WAVE EQ.)

- $u_t - u_{xx} = 0$ (HEAT EQ.)

- $u_{xx} + u_{yy} = 0$ (LAPLACE EQ.)

(e.g. EQ. FOR ELECTROSTATIC POTENTIAL)

WITH CHARGE DENSITY: $u_{xx} + u_{yy} = \rho(x,y)$

(POISSON EQ.)

- $u_t + u u_x - \varepsilon u_{xx} = 0$

(BURGERS' EQUATION)

RELEVANT IN FLUID DYNAMICS;

$u \equiv$ VELOCITY OF THE FLUID

- $u_t + c u u_x + u_{xxx} = 0$

(KORTEWEG-DE VRIES EQ.)

RELATED TO "SOLITONS"

- MAXWELL, NAVIER-STOKES, SCHRÖDINGER EQ'S...
- AND MORE...

USEFUL CONCEPTS

LINEAR PDE:

SAME CONCEPT AS FOR ODE'S.

THE PDE $L[u]=0$ IS LINEAR,

DIFFERENTIAL OPERATOR

WHEN $L[u_1]=L[u_2]=0$ IMPLIES

$L[u_1+u_2]=0$.

IN PRACTICE THIS MEANS L IS A LINEAR COMBINATION OF THE FORM

$$A(x_1,\dots x_n)\, u + \sum_i B^{(i)}(x_1,\dots,x_n)\, u_{x_i} + \sum_{i,j} C^{(i,j)}(x_1,\dots,x_n)\, u_{x_i x_j} + \dots = 0$$

COEFFICIENTS DO NOT DEPEND ON u

INHOMOGENEOUS "LINEAR" EQUATION:

$$L[u] = R(x_1, \dots, x_n)$$

THEN THE GENERAL SOLUTION IS GIVEN BY THE GENERAL SOLUTION OF THE HOMOGENEOUS PROBLEM, PLUS ANY PARTICULAR SOLUTION OF THE INHOMOGENEOUS ONE.

(THE SAME AS FOR ODE's).

LET US INTRODUCE TWO USEFUL DEFINITIONS.

* QUASI-LINEAR EQUATION: ONE THAT IS LINEAR WITH RESPECT TO THE HIGHEST DERIVATIVE TERMS.

e.g. THE BORN-INFELD PDE:

$$(1-u_t^2)\,u_{xx} + 2u_x u_t\, u_{xt} - (1+u_x^2)\,u_{tt} = 0$$

IS NONLINEAR, BUT "QUASI-LINEAR".

* FULLY NONLINEAR EQUATION: NONLINEAR WITH RESPECT TO THE HIGHEST ORDER DERIVATIVES.

e.g. THE EIKONAL EQUATION,

$$u_x^2 + u_y^2 = 1,$$

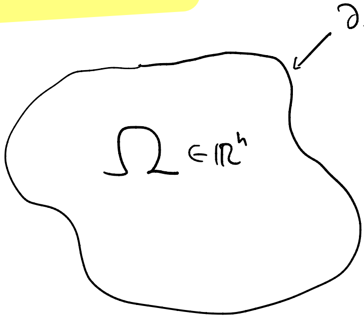
OR THE MONGE-AMPERE EQUATION,

$$u_{xy}^2 - u_{xx} u_{yy} = f(x,y),$$

ARE FULLY NONLINEAR.

WE WILL SEE THAT IN PDE's OFTEN THE MOST NATURAL PROBLEM IS NOT AN INITIAL VALUE PROBLEM BUT A BOUNDARY VALUE PROBLEM.

$\partial\Omega$ = BOUNDARY OF Ω



- PDE VALID INSIDE DOMAIN Ω
- BOUNDARY CONDITIONS ON u OR ITS DERIVATIVES AT THE BOUNDARY.

Ω: OPEN AND CONNECTED SET "DOMAIN"

THREE VERY IMPORTANT TYPES OF BOUNDARY CONDITIONS:

- **DIRICHLET BOUNDARY CONDITIONS**: THE VALUE OF u IS SPECIFIED AT POINTS OF $\partial\Omega$.

- **NEUMANN CONDITIONS**: THE VALUE OF THE NORMAL DERIVATIVE $\vec{n}\cdot\vec{\nabla} u$ IS SPECIFIED ALONG $\partial\Omega$

- **ROBIN BOUNDARY CONDITIONS**: THEY SPECIFY THE VALUE FOR A LINEAR COMBINATION OF u AND $\vec{n}\cdot\vec{\nabla} u$ ALONG $\partial\Omega$.

THE TYPICAL **INITIAL VALUE PROBLEM** FOR PDE'S IS CALLED $\boxed{\text{CAUCHY PROBLEM}}$:

→ FIND u SATISFYING THE EQUATION, SUCH THAT u AND ITS DERIVATIVES UP TO ORDER $n-1$ (WITH n = ORDER OF THE PDE) HAVE CERTAIN INITIAL VALUES ON A CERTAIN $(m-1)$-DIMENSIONAL SURFACE.

↳ FOR A PDE IN m VARIABLES.

A GENERAL SOLUTION TO ODE'S USUALLY DEPENDS ON A NUMBER OF CONSTANTS.

IN PDE'S, THE GENERAL SOLUTION HAS MORE DEGREES OF FREEDOM: IT MAY DEPEND ON UNFIXED FUNCTIONS.

LET US ILLUSTRATE THIS WITH A SIMPLE EXAMPLE.

FIND THE GENERAL SOLUTION FOR $u(x,y)$ SOLVING THE PDE: $u_x + u = e^{-x}$.

BECAUSE y IS "FROZEN", WE CAN SOLVE THIS AS AN ODE IN x. IT IS A SIMPLE, LINEAR, INHOMOGENEOUS ODE. A PARTICULAR SOLUTION IS

$$u_p(\cancel{x}) = x e^{-x}.$$

THE HOMOGENEOUS EQUATION IS SOLVED BY e^{-x}.

THE GENERAL SOLUTION IS THEREFORE:

$$u(x,y) = C(y)\, e^{-x} + x e^{-x}$$

THIS ARBITRARY CONSTANT NOW CAN BE AN ARBITRARY FUNCTION!

THE WAY TO SOLVE THIS EXAMPLE WILL BE VERY USEFUL LATER.

1st ORDER PDE's :

THE METHOD OF CHARACTERISTICS

FOR 1st ORDER PDE's, THERE IS A GENERAL METHOD TO REDUCE TO THE SOLUTION OF ODE's!

LET US START BY CONSIDERING A GENERAL **LINEAR 1st ORDER PDE** IN **TWO VARIABLES**

WE WILL GENERALISE TO MORE VARIABLES LATER

WE ASSUME WE ARE IN A REGION WITH $a(x,y) \neq 0$

$$\boxed{a(x,y)\, u_x + b(x,y)\, u_y = c(x,y)\, u + d(x,y)}$$

WE WANT TO FIND A CHANGE OF VARIABLES $(x,y) \to (\xi, \eta)$ TO GO TO THE CANONICAL FORM:

SIMPLE TO SOLVE BECAUSE η IS "FROZEN"

$$\boxed{A(\xi,\eta)\, u_\xi = C(\xi,\eta)\, u + D(\xi,\eta)},$$

(WHERE $A(\xi,\eta) = a(x(\xi,\eta), y(\xi,\eta))$,
$C(\xi,\eta) = c(x(\xi,\eta), y(\xi,\eta))$, etc...)

TO FIND THE APPROPRIATE ξ, η, NOTICE THAT:

$$u_x = \xi_x u_\xi + \eta_x u_\eta$$

$$u_y = \xi_y u_\xi + \eta_y u_y .$$

THE PDE, THEN, FROM THE ORIGINAL FORM, CAN BE REWRITTEN AS

$$a \cdot u_x + b u_y = c u + d$$

$$\Longrightarrow \boxed{u_\xi \cdot (a \xi_x + b \xi_y) + u_\eta \underline{(a \eta_x + b \eta_y)} = c \cdot u + d}$$

<u>IN ORDER TO GO TO THE CANONICAL FORM, WE WANT TO IMPOSE</u>:

$$\boxed{a \eta_x + b \eta_y = 0}$$

THIS IS ALSO A 1st ORDER LINEAR PDE, BUT IT IS SIMPLER TO SOLVE.

WE NOTICE THAT IT CAN BE INTERPRETED AS THE CONDITION

$$\boxed{\frac{d}{dx} \eta(x, y(x)) = 0}$$

, WHERE $\underline{y(x)}$ IS A SOLUTION OF:

$$\boxed{\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}} \quad (*)$$

THIS ODE DEFINES THE CURVES $y(x)$, CALLED CHARACTERISTIC CURVES. THE FUNCTION $\eta(x,y)$ SHOULD BE CONSTANT ALONG EACH CHARACTERISTIC CURVE: THIS MEANS THAT, PRACTICALLY, WE CAN IDENTIFY $\eta(x,y)$ WITH THE INTEGRATION CONSTANT ARISING IN THE SOLUTION OF THE ODE $(*)$.

(WE WILL SEE HOW THIS WORKS SHORTLY)

THE CHARACTERISTIC CURVES ARE GIVEN BY:

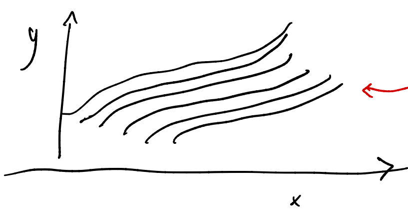
$$\eta(x,y) = K$$

CONSTANT

WE ASSUME WE ARE AROUND A POINT WHERE $\eta_y \neq 0$

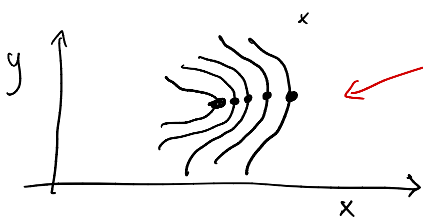
WHICH MEANS DIFFERENT CHARACTERISTIC CURVES DO NOT "FOLD" AND WE CAN HAVE A 1:1 MAP $(x,y) \to (\xi,\eta)$ $\xi \equiv x$.

(1)



CHARACTERISTIC CURVES WITH $\eta_y \neq 0$

(2)



CHARACTERISTIC CURVES WITH $\eta_y = 0$ AT THE MARKED POINTS

SINCE WE ASSUMED $a(x,y) \neq 0$, WE ARE IN THE FIRST SITUATION (FIG. $\boxed{(1)}$) WITH $\eta_y \neq 0$.

THEN WE CAN TAKE COORDINATES:

$$\begin{cases} \xi = x \\ \eta = \eta(x,y) \end{cases}$$

$\eta(x,y)$ ← OBTAINED BY SOLVING FOR THE CHARACTERISTIC CURVES.

THIS CHANGE OF VARIABLE IS LOCALLY 1:1

IN FACT THE JACOBIAN $J = \left| \frac{\partial(\xi,\eta)}{\partial(x,y)} \right| = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \eta_y \neq 0$

THE PDE BECOMES:

$$a\, u_x + b\, u_y = c\, u + d$$

$$\hookrightarrow \quad A(\xi,\eta)\, u_\xi = C(\xi,\eta) \cdot u + d(\xi,\eta)$$

WHICH CAN BE SOLVED AS IN THE EXAMPLE ABOVE: IT IS AN ODE IN ξ, AND INTEGRATION CONSTANTS CAN BE ARBITRARY FUNCTIONS OF η.

EXAMPLE.

$$x u_x - y u_y + y^2 u = y^2$$

$x, y \neq 0$

IN THE NOTATION ABOVE

$$a(x,y) = x$$
$$b(x,y) = -y$$
$$c(x,y) = -y^2$$
$$d(x,y) = y^2$$

LET US FIND $\eta(x,y)$ SUCH THAT $a\,\eta_x + b\,\eta_y = 0$

THE CHARACTERISTIC CURVES SOLVE:

$$\frac{dy}{dx} = \frac{b}{a} = -\frac{y}{x}$$

THIS IS A SEPARABLE ODE WITH SOLUTION:

$$\log y = -\log x + C$$

$$\rightarrow \quad x \cdot y = K \leftarrow \text{const.}$$

SINCE $\eta(x,y)$ SHOULD BE CONSTANT ON CHARACTERISTIC CURVES WE CAN CHOOSE

$$\boxed{\eta(x,y) = x \cdot y}$$

THIS SATISFIES $\eta_y = x \neq 0$ FOR $x \neq 0$ ✓

THEN WE CHANGE VARIABLES:

$$(x,y) \to (\xi, \eta) = (x, xy)$$

THE COEFFICIENTS OF THE PDE TRANSFORM AS:

$$a(x,y) = x = \xi$$

$b(x,y) = \dots$ (WE WON'T NEED IT, THIS TERM CANCELS!)

$$c(x,y) = -y^2 = -\frac{\eta^2}{\xi^2}.$$

$$d(x,y) = y^2 = \eta^2/\xi^2.$$

THE PDE BECOMES:

$$\xi u_\xi = \frac{\eta^2}{\xi^2} \cdot (-u + 1)$$

THE HOMOGENEOUS EQUATION $\xi u_\xi + \frac{\eta^2}{\xi^2} u = 0$ IS SEPARABLE:

$$\hookrightarrow \frac{u_\xi}{u} = -\frac{\eta^2}{\xi^3}$$

$$\rightarrow u = C(\eta)\, e^{\frac{\eta^2}{2\xi^2}}$$

NOTICE THE INTEGRATION "CONSTANT" WHICH IS FUNCTION OF η!

WE STILL NEED A PARTICULAR SOLUTION OF THE INHOMOGENEOUS EQUATION

WE NOTICE THAT $u = 1$ IS A SOLUTION OF

$$\xi u_\xi + \frac{\eta^2}{\xi^2}(u-1) = 0$$

THEREFORE, THE GENERAL SOLUTION OF THE PDE IS :

$$u(\xi, \eta) = 1 + C(\eta)\, e^{\frac{\eta^2}{2\xi^2}}$$

NAMELY,

$$\boxed{u(x,y) = 1 + C(xy)\, e^{\frac{y^2}{2}}}$$

EXERCISES:

FIND THE GENERAL SOLUTION OF:

* $x u_x + y u_y = \gamma \cdot u$

* $a u_x + b u_y = c u + d$, a, b, c, d CONSTANTS WITH $a^2 + b^2 \neq 0$