EXAMPLE: LET US SOLVE THE CAUCHY

PROBLEM:

$$u\, u_x + u_y = 1/2$$

WITH INITIAL CONDITION $u(s,s) = \frac{s}{4}$

FOR $s \in [0,1]$.

WE ARE IN THE CONDITIONS OF THE THEOREM

$$\begin{vmatrix} a & b \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} s/4 & 1 \\ 1 & 1 \end{vmatrix} = \frac{s}{4} - 1 \neq 0 \quad \text{FOR } s \in [0,1]$$

CHARACTERISTICS ODE:

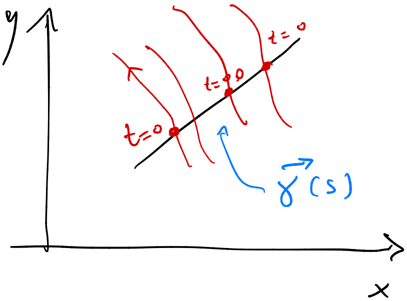
$$a(x,y,u) = u$$

$$b(x,y,u) = 1$$

$$c(x,y,u) = \frac{1}{2}$$

$$\rightarrow \begin{cases} \dfrac{dx}{dt} = u \\[2ex] \dfrac{dy}{dt} = 1 \\[2ex] \dfrac{du}{dt} = 1/2 \end{cases}$$

WE WILL CHOOSE THE PARAMETER t SUCH THAT $t=0$ ARE POINTS ON THE INITIAL CURVE.



THEREFORE, WE LOOK FOR SOLUTIONS

$$x(s,t) \,,\; y(s,t) \,,\, u(s,t)$$

WITH INITIAL CONDITIONS

$$x(s,t)\Big|_{t=0} = x_0(s) = s$$

$$y(s,t)\Big|_{t=0} = y_0(s) = s$$

$$u(s,t)\Big|_{t=0} = u_0(s) = \frac{s}{4}$$

THE SOLUTION OF THE ODE'S WITH THESE INITIAL CONDITIONS IS:

$$\frac{du}{dt} = \frac{1}{2} \longrightarrow u(s,t) = \frac{s}{4} + \frac{t}{2}$$

$$\frac{dx}{dt} = u \longrightarrow x(s,t) = s + \frac{ts}{4} + \frac{t^2}{4}$$

$$\frac{dy}{dt} = 1 \longrightarrow y(s,t) = s + t$$

NOW WE CAN INVERT:

$$(s,t) \longrightarrow (x,y)$$

LOCALLY, THIS IS INVERTIBLE 1:1 BECAUSE

$$\begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} = \begin{vmatrix} \dot{x}_0(s) & a \\ \dot{y}_0(s) & b \end{vmatrix} \neq 0$$

(THE USUAL CONDITION)

IN OUR CASE WE FIND EXPLICITLY

$$y = s + t$$

$$x = \frac{t^2}{4} + s + \frac{st}{4} \longrightarrow \begin{cases} s = \dfrac{4x - y^2}{4 - y} \\[2ex] t = \dfrac{4y - 4x}{4 - y} \end{cases}$$

AND FINALLY:

$$\boxed{u = \frac{1}{2} t + \frac{s}{4} = \frac{\left(2y - x - \frac{y^2}{4}\right)}{4 - y}}$$

LET US STUDY ANOTHER SIMPLE EXAMPLE

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$$\boxed{y\, u_x - x\, u_y = 0}$$

CHARACTERISTIC CURVES

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x \\ \dfrac{du}{dt} = 0 \end{cases}$$

SOLUTION

$$x(t) = \cos(t - t_0)\, K$$
$$y(t) = \sin(t - t_0)\, K$$

$$\hookrightarrow \quad x^2 + y^2 = \text{const.}$$

$$u(x,y) = F(x^2 + y^2)$$

GENERAL SOLUTION

WITH F : GENERIC FUNCTION

* LET US NOW CONSIDER THE CAUCHY PROBLEM DEFINED BY:

$$u(\cos s, \sin s) = 1 \quad , \quad s \in [0, \tfrac{\pi}{2}]$$

SINCE $\vec{\gamma}$ IS PARALLEL TO A CHARACTERISTIC CURVE, IT VIOLATES ASSUMPTIONS OF THE THEOREM:

$$\begin{vmatrix} \dot{x}(s) & \dot{y}(s) \\ a & b \end{vmatrix} = \begin{vmatrix} -\sin s & \cos s \\ \sin s & -\cos s \end{vmatrix} = 0$$

IN THIS CASE THE PROBLEM HAS INFINITELY MANY SOLUTIONS. IN FACT, THE INITIAL CONDITION IS SATISFIED BY ANY FUNCTION

$u = F(x^2+y^2)$ AS LONG AS $\underline{F(1) = 1}$

* INSTEAD, IF WE CONSIDER THE INITIAL CONDITION; e.g.:

$$u(\cos s, \sin s) = \cos^2 s$$

THE CAUCHY PROBLEM HAS NO SOLUTION SINCE THERE IS NO CHOICE OF F WHICH CAN SATISFY THIS CONDITION.

* OF COURSE, THESE PATHOLOGIES WERE AN EFFECT OF THE CHOICE OF $\vec{\gamma}$ ON WHICH WE GAVE THE INITIAL CONDITION.