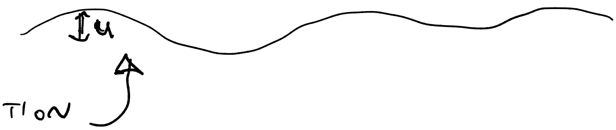
WAVE EQUATION

LET'S START BY CONSIDERING THE WAVE EQ. IN ONE SPACE DIMENSION (PLUS TIME):

$$\boxed{u_{tt} - c^2 u_{xx} = 0}$$

THIS EQ. DESCRIBES e.g. VIBRATIONS OF A STRING

WHERE



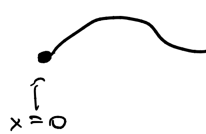
u = HEIGHT OF OSCILLATION

WE WILL CONSIDER VARIOUS TYPES OF IVP (INITIAL VALUE PROBLEMS)

AND BIVP (BOUNDARY INITIAL VALUE PROBLEMS)

THE TWO MAIN TYPES OF BOUNDARY CONDITIONS ARE:

$u(0,t) = 0$ (DIRICHLET) $\Longrightarrow$ END OF STRING IS FIXED AT $x=0$.



$u_x(0,t) = 0$ (NEUMANN) $\Longrightarrow$ NO FORCE ON THE STRING END POINT AT $x=0$

GENERAL SOLUTION

BECAUSE $\left(\partial_t^2 - c^2 \partial_x^2\right) = \left(\partial_t - c\,\partial_x\right)\left(\partial_t + c\,\partial_x\right)$

WE CAN ALWAYS DECOMPOSE THE SOLUTION OF THE 1D WAVE EQUATION INTO TWO SOLUTIONS OF TRANSPORT EQUATION WITH SPEED $+c$ AND $-c$.

THE GENERAL SOLUTION TO $\left(\partial_t^2 - c^2 \partial_x^2\right) u = 0$ IS:

$$\boxed{u(x,t) = F(x-ct) + G(x+ct)}$$

TO SOLVE A SPECIFIC PROBLEM (IVP o BIVP) WE NEED TO CHOOSE APPROPRIATELY F AND G.

WE WILL SEE:

- D'ALEMBERT FORMULA : SOLUTION TO CAUCHY PROBLEM ON THE INFINITE LINE
- METHOD OF IMAGES TO SOLVE IBVP WITH HOMOGENEOUS BOUNDARY CONDITIONS
- MORE GENERAL METHOD: FOURIER DECOMPOSITION

• HIGHER DIMENSIONS.

D'ALEMBERT FORMULA

CONSIDER THE I.V.P.

$$u_{tt} - c^2 u_{xx}, \quad t \geq 0, \quad x \in \mathbb{R}$$

WITH
$$\boxed{\begin{cases} u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}} \leftarrow \text{I.C. (INITIAL CONDITION)}$$

NOTICE THAT, SINCE THE PDE IS 2nd ORDER IN t, THE INITIAL CONDITION HAS TO INCLUDE THE VALUE OF u_t AT $t=0$.

WE WILL SOLVE THE PROBLEM BY SOLVING

F, G SUCH THAT

$$u(x,t) = F(x-ct) + G(x+ct).$$

IMPOSING THE I.C. :

$$\begin{cases} F(x) + G(x) = \varphi(x) \\ -c \cdot \left(F'(x) - G'(x)\right) = \psi(x) \end{cases}$$

INTEGRATING THE SECOND RELATION WE FIND:

$$F(x) - G(x) = -\frac{1}{c} \int_{x_0}^{x} \psi(s)\, ds$$

TOGETHER WITH THE FIRST RELATION, WE HAVE:

$$F(x) = \frac{1}{2}\left(\varphi(x) - \frac{1}{c}\int_{x_0}^{x} \psi(s)\, ds\right)$$

$$G(x) = \frac{1}{2}\left(\varphi(x) + \frac{1}{c}\int_{x_0}^{x} \psi(s)\, ds\right)$$

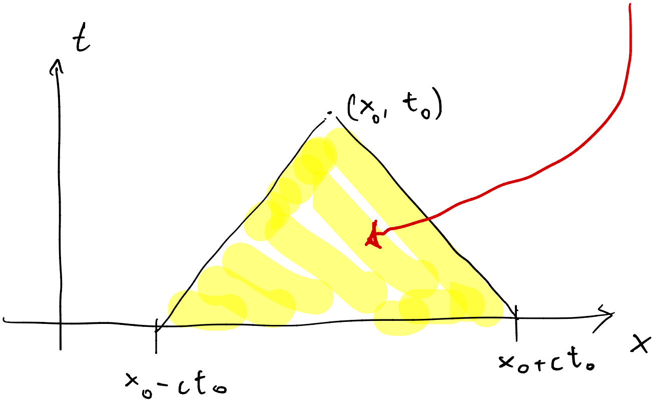
FROM $u = F(x-ct) + G(x+ct)$, WE FINALLY HAVE:

$$\boxed{u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds}$$

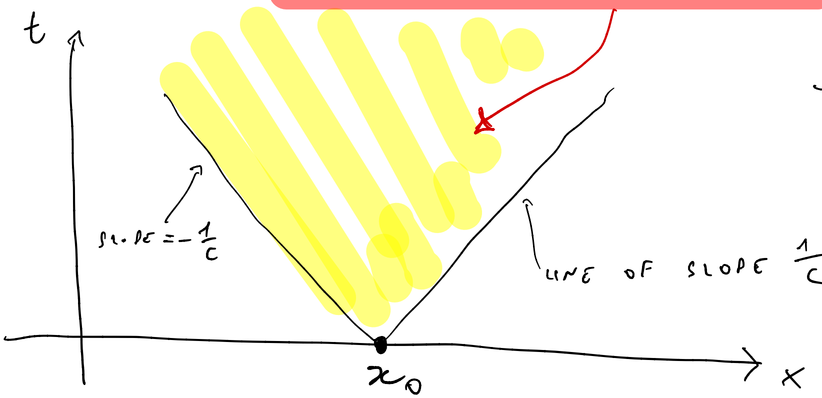
THIS IS D'ALEMBERT'S FORMULA

D'ALEMBERT FORMULA GIVES THE SOLUTION OF THE I.V.P. ON THE LINE.

NOTICE THAT THE SOLUTION ~~AT~~ (x_0, t_0) DEPENDS ON THE INITIAL DATA INSIDE A FINITE DOMAIN OF DEPENDENCE:



SIMILARLY, A POINT x_0 ON THE REAL AXIS CAN ONLY AFFECT THE SOLUTION INSIDE A CERTAIN DOMAIN OF INFLUENCE (FINITE FOR ANY TIME t)



THIS MEANS THAT SIGNALS PROPAGATE WITH FINITE SPEED.

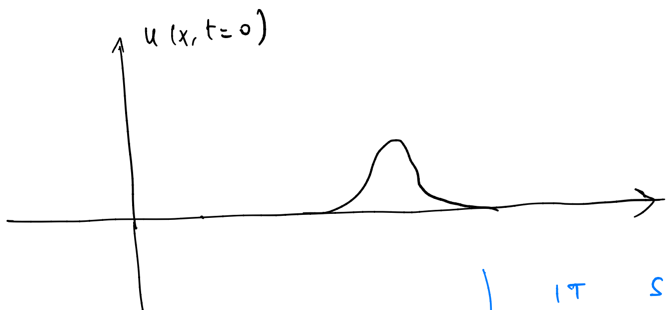
(IT WILL BE DIFFERENT FOR OTHER EQ'S!)

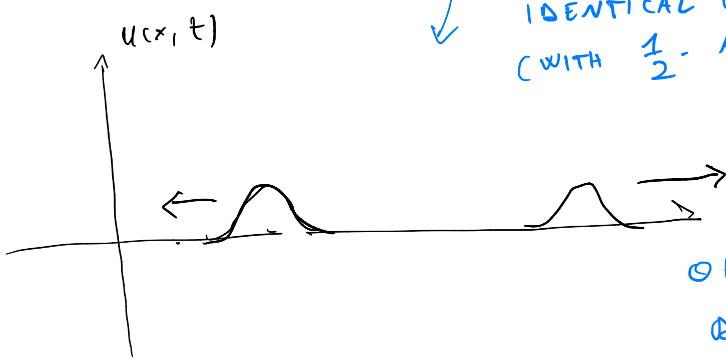
NOTE: IT IS EASY TO SEE WHAT HAPPENS WHEN THE INITIAL CONDITION HAS $\psi = 0$

THEN D'ALEMBERT'S FORMULA BECOMES:

$$u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + 0 .$$

WHERE $\varphi(x)$ IS THE INITIAL PROFILE.





IT SPLITS INTO TWO IDENTICAL IMPULSES (WITH $\frac{1}{2}$ AMPLITUDE) TRAVELLING WITH SPEED C IN OPPOSITE DIRECTIONS.

SOLUTION TO INHOMOGENEOUS WAVE EQUATION ON THE LINE

$$u_t - c^2 u_{xx} = f(x,t) \longleftarrow$$ EXTERNAL DRIVING FORCE

I.C. $\begin{cases} u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$

WE CAN SOLVE THE PROBLEM BY DIVIDING IT IN TWO:

$$u = \underline{u_1} + \underline{u_2}$$ WHERE:

- u_1 SOLVES THE HOMOGENEOUS PROBLEM WITH THE INITIAL DATA OF THE FULL PROBLEM

$$(\partial_{tt} - c^2 \partial_{xx})\, u_1 = 0$$

$$\begin{cases} u_1(x,0) = \varphi(x) \\ \partial_t u_1(x,t)\big|_{t=0} = \psi(x) \end{cases}$$

- u_2 SOLVES THE <u>INHOMOGENEOUS PROBLEM</u> WITH INITIAL DATA $= 0$:

$$(\partial_{tt} - c^2 \partial_{xx})\, u_2 = f(x,t)$$

$$\begin{cases} u_2(x,0) = 0 \\ \partial_t u_2(x,t)\big|_{t=0} = 0 \end{cases}$$

THIS IS A COMMON STRATEGY. WE WILL SEE IT AGAIN.

- u_1 IS GIVEN BY D'ALEMBERT'S FORMULA
- u_2 CAN BE FOUND EXPLICITLY USING THE

CHANGE OF VARIABLES:

$$(x,t) \longrightarrow (\xi,\eta) = (x+ct,\ x-ct)$$

$$\hookrightarrow \quad (\partial_t^2 - c^2\partial_x^2) = \frac{1}{4c^2}\partial_\xi\partial_\eta$$

$$\hookrightarrow \quad \partial_\eta\partial_\xi u_2 = 4c^2 \cdot f \qquad (*)$$

LET US DENOTE THE SOLUTION REWRITTEN AS F'N OF ξ,η

AS: $U(\xi,\eta) = u_2\left(\frac{1}{2}(\eta+\xi),\ \frac{1}{2c}(\xi-\eta)\right)$

THE INITIAL CONDITION $u_2(x,0) = 0$

BECOMES: $U(\xi,\xi) = 0$

WHILE $\partial_t u_2(x,t)\Big|_{t=0} = 0$

FURTHER IMPLIES:

$$\partial_\xi U(\xi,\xi) = \partial_\eta U(\xi,\xi) = 0$$

SOLVING THE PDE

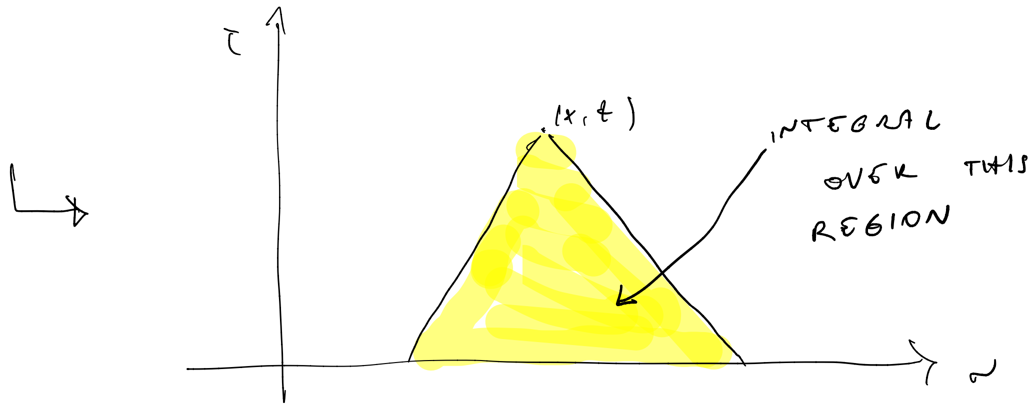
$$(*) \qquad \frac{1}{4c^2}\, \partial_\xi \partial_\eta U = F(\xi,\eta)$$

WITH THESE INITIAL CONDITIONS WE FIND:

$$U(\xi,\eta) = -4c^2 \int_\eta^\xi d\lambda \int_\eta^\lambda F(\lambda,\rho)\, d\rho$$

GOING BACK TO THE ORIGINAL VARIABLES:

$$u_2(x,t) = \frac{1}{2c} \cdot \int_0^t d\tau \int_{x-|c|(t-\tau)}^{x+|c|(t+\tau)} d\sigma \; f(\sigma,\tau)$$



INITIAL - BOUNDARY VALUE PROBLEMS

LET US NOW INTRODUCE BOUNDARIES

i.e. WE WANT TO STUDY THE WAVE EQUATION ON HALF - LINE, OR ON A FINITE INTERVAL.

- WE FIRST STUDY **HOMOGENEOUS BOUNDARY CONDITIONS**

→ IN THIS CASE WE CAN USE THE

$\boxed{\text{METHOD OF IMAGES}}$

(OR, METHOD OF REFLECTION)

- AFTER THAT, WE WILL STUDY MORE GENERAL BOUNDARY CONDITIONS.

IN THIS CASE THE I.B.V.P. CAN BE SOLVED WITH THE **FOURIER METHOD**

(OR "EIGENFUNCTIONS DECOMPOSITION METHOD")

BOTH THESE METHODS WILL HAVE IMPORTANT GENERALISATIONS TO PROBLEMS IN HIGHER DIMENSIONS, AND WILL BE APPLIED ALSO FOR THE HEAT EQUATION AND THE LAPLACE EQUATION.

FIRST WE CONSIDER HOMOGENEOUS BOUNDARY CONDITIONS.

DEF. A BOUNDARY CONDITION IS HOMOGENEOUS WHEN THE LINEAR COMBINATION OF TWO FUNCTIONS SATISFYING THE B.C., WILL ALSO SATISFY THE SAME B.C.

EXAMPLE: DIRICHLET OF THE TYPE

$$u(0,t) = 0$$

OR NEUMANN OF THE TYPE $\partial_x u(0,t) = 0$

NOTE: INSTEAD, B.C. $u(0,t) = K$ IS NOT HOMOGENEOUS FOR $K \neq 0$.

METHOD OF IMAGES

LET US CONSIDER A PROBLEM ON THE HALF LINE

$$\left(\partial_t^2 - c^2 \partial_x^2\right) u(x,t) = 0 \quad , \qquad x > 0.$$

I. C. $\begin{cases} u(x,0) = \varphi(x) \;, & x > 0 \\ u_t(x,0) = \psi(x) \end{cases}$

WE START BY CONSIDERING DIRICHLET B.C. AT $x = 0$ (HOMOGENEOUS)

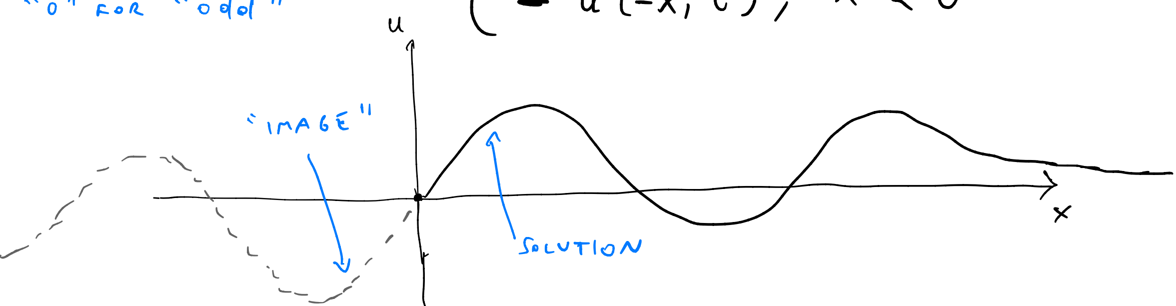
BC: $u(0,t) = 0$

THEN WE CAN USE THE FOLLOWING TRICK.

CONSIDER THE ODD EXTENSION OF THE SOLUTION

$$u_o(x,t) = \begin{cases} u(x,t) \;, & x > 0 \\ -u(-x,t) \;, & x < 0 \end{cases}$$

"o" FOR "odd"



WE WANT TO SEE $u_{odd}(x,t)$ AS A SOLUTION OF THE WAVE EQUATION ON THE WHOLE LINE.

ITS INITIAL CONDITIONS ARE:

$$\begin{cases} u_o(x,0) = \varphi_o(x), & x \in \mathbb{R} \\ \partial_t u_o(x,t)\Big|_{t=0} = \psi_o(x), & x \in \mathbb{R}. \end{cases} \qquad \boxed{(I.C.)}$$

WHERE THE ODD EXTENSIONS OF THE INITIAL DATA ARE DEFINED AS ABOVE.

FOR A GENERIC FUNCTION ON $\mathbb{R}^+$

$$f(x) \to f_o(x) = \begin{cases} f(x), & x > 0 \\ -f(-x), & x < 0 \end{cases}$$

SO WE:

- SOLVE THE PROBLEM ON THE INFINITE LINE (WITH NO BOUNDARIES) WITH THE INITIAL DATA $\boxed{(I.C.)}$

- RESTRICT THE SOLUTION TO $x > 0$ TO SOLVE THE ORIGINAL PROBLEM

THE FIRST STEP IS DONE WITH THE D'ALEMBERT FORMULA.

IN CONCLUSION, WE FIND:

$$u_0(x,t) = \frac{1}{2}\Big(\varphi_0(x+ct) + \varphi_0(x-ct)\Big) + \frac{1}{2c}\cdot\int_{x-ct}^{x+ct} \psi_0(s)\,ds$$

RESTRICTING TO $x > 0$, WE CAN WRITE THE SOLUTION ON THE INFINITE LINE AS:

FOR $x > ct$:

$$u(x,t) = \frac{1}{2}\Big(\varphi(x+ct) + \varphi(x-ct)\Big) + \frac{1}{2}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

FOR $x < ct$:

$$u(x,t) = \frac{1}{2}\Big(\varphi(x+ct) - \varphi(ct-x)\Big) + \frac{1}{2}\int_{ct-x}^{x+ct} \psi(s)\,ds$$

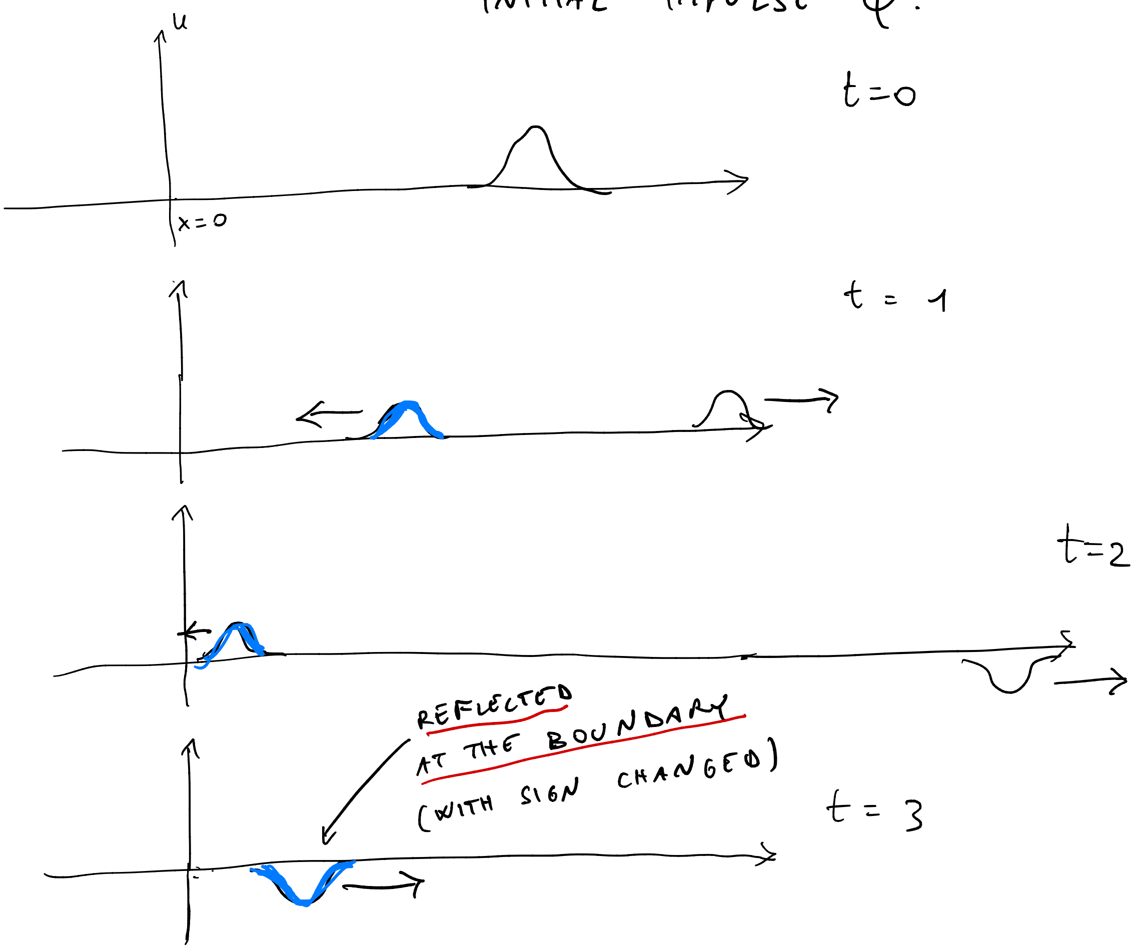
(WE ASSUME $c > 0$)

NOTE: INDEED WE CAN CHECK THAT $u(0,t) = 0$

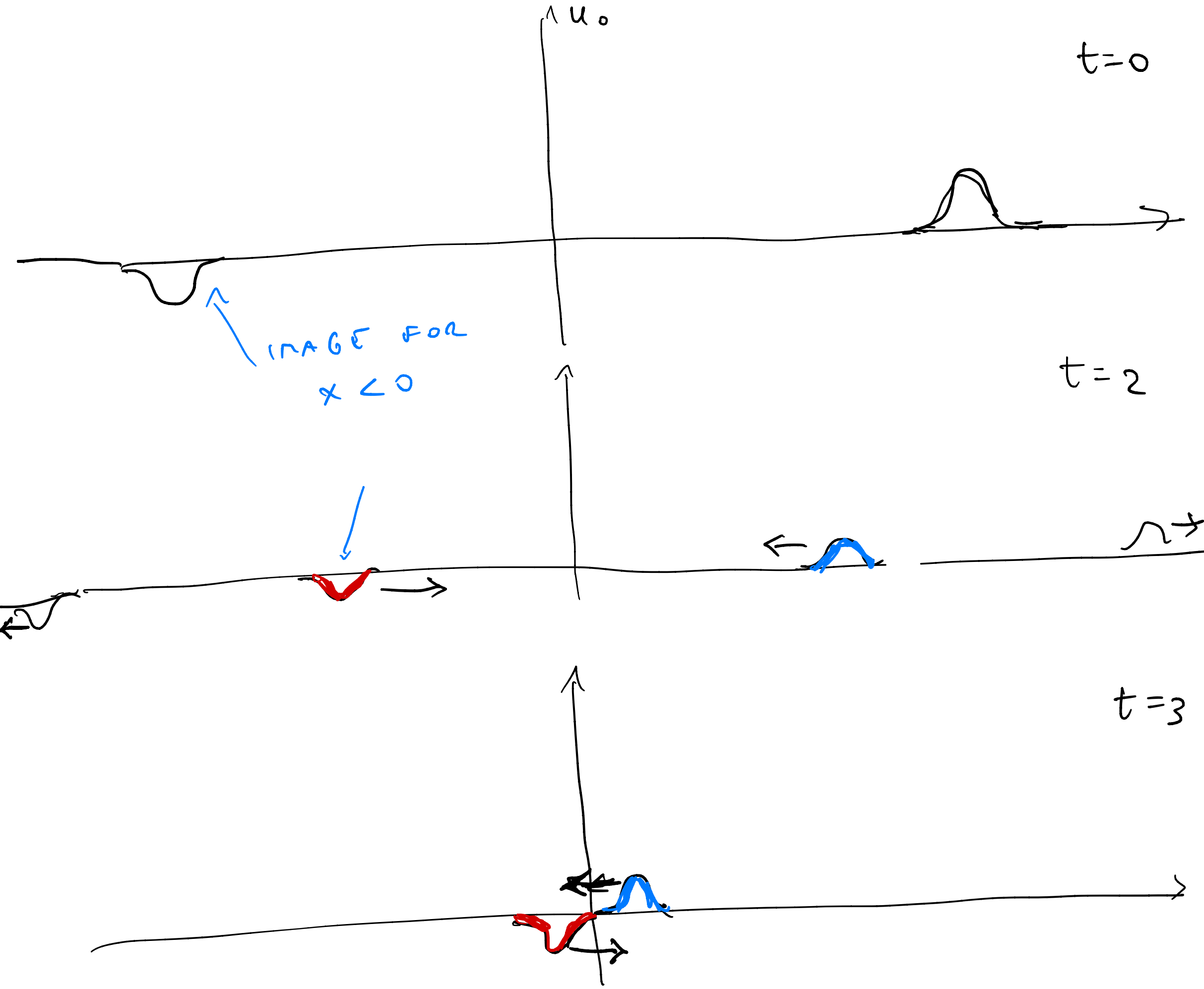
→ CHECK:

$$u(0,t) = \frac{1}{2}\left(\cancel{\varphi(0+ct) - \varphi(ct-0)} \right) + \frac{1}{2} \cancel{\int_{ct-0}^{ct+0} \psi(s)\, ds} = 0$$

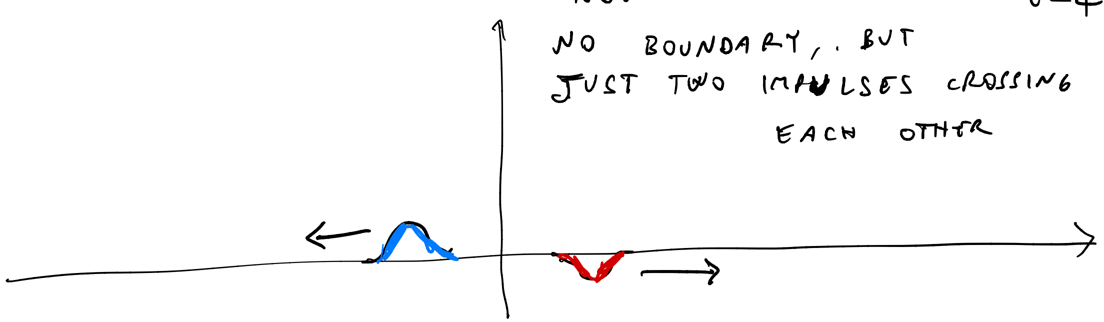
IN FIGURES: LET US ILLUSTRATE THE CASE WITH $\psi = 0$, WITH A LOCALISED INITIAL IMPULSE φ.



THE SAME DYNAMICS IS REPRODUCED AS FOLLOWS BY THE METHOD OF IMAGES



NOW THERE IS NO BOUNDARY, BUT JUST TWO IMPULSES CROSSING EACH OTHER



DIFFERENT INTERPRETATIONS, GIVING EXACTLY THE SAME DYNAMICS FOR $x>0$.

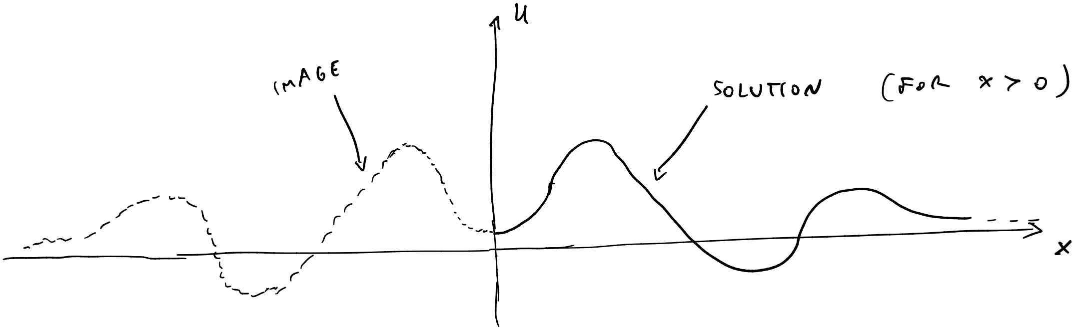
LET US NOW CONSIDER THE PROBLEM ON THE HALF LINE WITH NEUMANN B.C.'s.

PDE: $u_{tt} - c^2 u_{xx} = 0 \quad , \quad x>0$

IC: $\begin{cases} u(x, t=0) = \varphi(x) \\ u_t(x,t)\big|_{t=0} = \psi(x) \end{cases}$

B.C. $u_x(x,t)\big|_{x=0} = 0$

IN THIS CASE THE SOLUTION IS FOUND BY TAKING THE "EVEN" EXTENSION OF THE SOLUTION, FROM THE HALF LINE TO THE FULL LINE



$$u_e(x,t) = \begin{cases} u(x,t), & x>0 \\ +u(-x,t), & x<0 \end{cases}$$

"e" FOR "even"

CONSIDER THE SAME EXTENSION FOR THE INITIAL DATA:

$\downarrow$

$$\varphi_e(x,t) = \begin{cases} \varphi(x,t), & x>0 \\ \varphi(-x,t), & x<0 \end{cases} \quad , \quad \psi_e(x,t) = \begin{cases} \psi(x,t), & x>0 \\ \psi(-x,t), & x<0 \end{cases}$$

NOW, AS BEFORE, WE SOLVE THE PROBLEM ON THE FULL-LINE WITH THESE INITIAL DATA, AND RESTRICT TO $x>0$ TO RECOVER $u(x,t)$.

THE SOLUTION FOR $u_e(x,t)$ IS: (D'ALEMBERT FORMULA):

$$u_e(x,t) = \frac{1}{2}\left(\varphi_e(x+ct) + \varphi_e(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_e(s)\,ds$$

THIS MEANS THAT

FOR $x > ct$: $$u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

FOR $0 < x < ct$: $$u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)\,ds + \frac{2}{2c}\cdot\int_{0}^{ct-x} \psi(s)\,ds$$

* <u>EXERCISE</u>: CHECK THAT THIS SOLUTION AUTOMATICALLY GUARANTEES $u_x = 0$ AT $x = 0$.

NOTICE THAT, PROVIDED $\varphi(x)$ AND $\psi(x)$ ARE CONTINUOUS, THIS CONDITION HOLDS, FOR $t>0$, EVEN WHEN AT $t=0$ IT IS VIOLATED, i.e. WHEN $\varphi_x\big|_{x=0} \neq 0$.

CASE OF AN INTERVAL

WE CAN USE THE METHOD OF IMAGES ALSO TO TREAT THE BIVP ON AN INTERVAL WITH HOMOGENEOUS BOUNDARY C'S.

IN THIS CASE WE BUILD A PERIODIC EXTENSION OF THE SOLUTION ON THE INTERVAL WITH THE FOLLOWING RULES:

- THE SOLUTION SHOULD BE ODD ACROSS EVERY BOUNDARY WITH (HOMOGENEOUS) DIRICHLET B.C. (i.e. $u = 0$)

- THE SOLUTION SHOULD BE EVEN ACROSS EVERY BOUNDARY WITH (HOMOGENEOUS) NEUMANN B.C. (i.e, $u_x = 0$)

[TO BE CONTINUED...]