In[9]:=

```
α = 1;
```

Problema Dirichlet (bordi a temperatura fissata = 0)

Problem: a bar of length L = 1, has initial temperature distribution at t=0 given by u(x,0) = 50 x (1-x), x in [0,1]. The two ends of the bar are kept at temperature = 0. Find the temperature distribution at t>0 . Assume the diffusivity coefficient is $\alpha = 1$.

To solve this problem we use the Fourier method.

```
(* Compute Fourier coefficients *)
Integrate[ x (1 - x) Sin[ n Pi x], {x, 0, 1}] 2 ;
```

Out[∘]=

$$\frac{2\ (2 - 2\ \text{Cos}[n\,\pi] - n\,\pi\ \text{Sin}[n\,\pi])}{n^3\,\pi^3}$$

In[5]:=

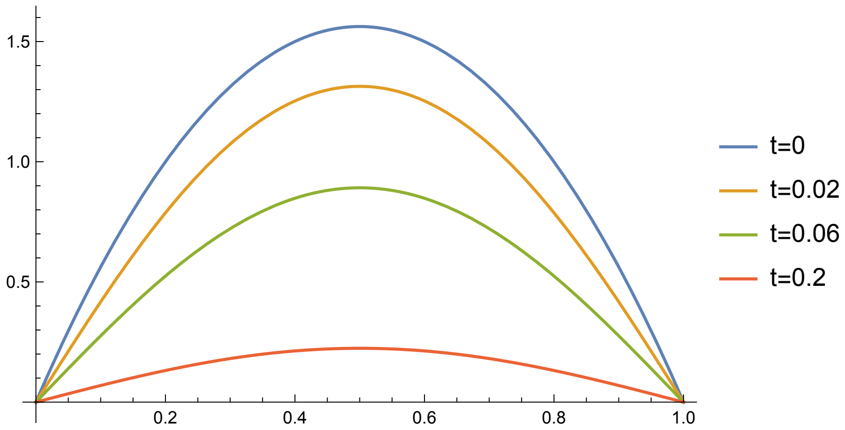
```
Clear[F]
F[x_, t_][NN_] := 50 Sum[ Sin[n Pi x] E^(- n^2 Pi^2 α t) / Pi^3 / n^3, {n, 1, NN, 2}]
```

In[10]:=

```
Print["Solution for different times:"]
Quiet[Plot[{ F[x, 0.0][200],  F[x, 0.02][200],  F[x, 0.06][200], F[x, 0.2][200]} ,
  {x, 0, 1}, PlotLegends → {"t=0", "t=0.02", "t=0.06", "t=0.2"} ]]
```

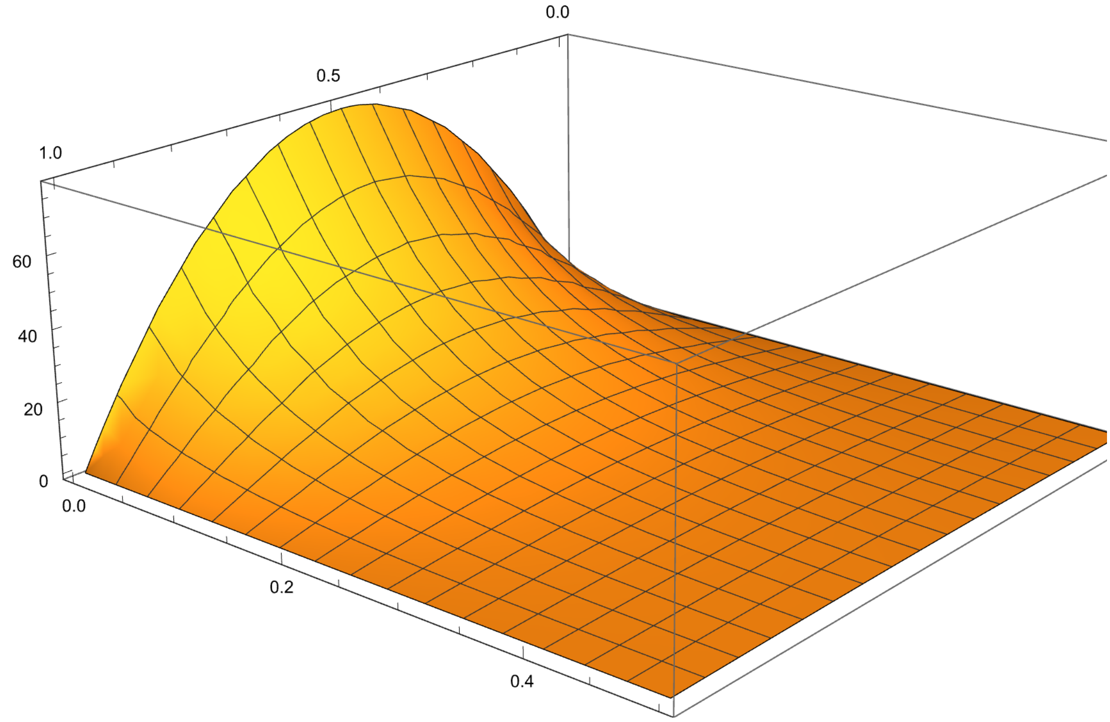
Solution for different times:

Out[11]=



In[◦]:=
```
Plot3D[50 F[x, t][200], {x, 0, 1}, {t, 0, 1/2}, PlotRange → All]
```

Out[◦]=



Problema con condizione iniziale con un punto di non differenziabilita`

Problem: We consider the same parameters as above, but now the initial condition is a sharp triangle: u(x,0) = x for x in [0,1/2] and (1-x) for x in [1/2, 1].
We can still use the Fourier method in the same way. Notice that the Fourier series captures well solutions with isolated points of non differentiability.

In[12]:=
```
(* Compute Fourier coefficients *)
2 (Integrate[ Sin[n Pi x] x, {x, 0, 1/2}] +
   Integrate[ Sin[n Pi x] (1 - x), {x, 1/2, 1}] )
```

Out[12]=

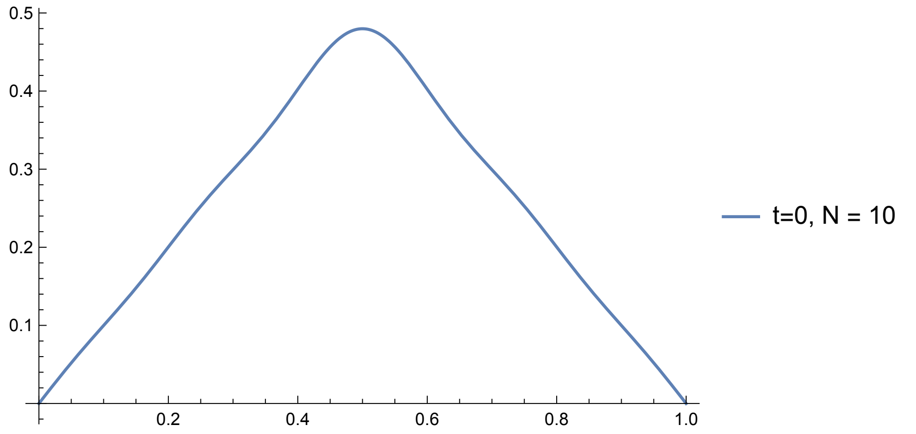
$$2\left(\frac{-\frac{1}{2}\,n\,\pi\,\mathrm{Cos}\left[\frac{n\pi}{2}\right]+\mathrm{Sin}\left[\frac{n\pi}{2}\right]}{n^2\,\pi^2}+\frac{n\,\pi\,\mathrm{Cos}\left[\frac{n\pi}{2}\right]+2\,\mathrm{Sin}\left[\frac{n\pi}{2}\right]-2\,\mathrm{Sin}[n\,\pi]}{2\,n^2\,\pi^2}\right)$$

In[13]:=
```
Clear[F4]
F4[x_, t_][NN_] := Sum[ Sin[n Pi x] E^(- n^2 Pi^2 α t) 2
```
$$\left(\frac{-\frac{1}{2}\, n\,\pi\, \mathrm{Cos}\left[\frac{n\pi}{2}\right] + \mathrm{Sin}\left[\frac{n\pi}{2}\right]}{n^2\,\pi^2} + \frac{n\,\pi\,\mathrm{Cos}\left[\frac{n\pi}{2}\right] + 2\,\mathrm{Sin}\left[\frac{n\pi}{2}\right] - 2\,\mathrm{Sin}[n\,\pi]}{2\,n^2\,\pi^2}\right),\ \{n, 1, NN, 1\}]$$

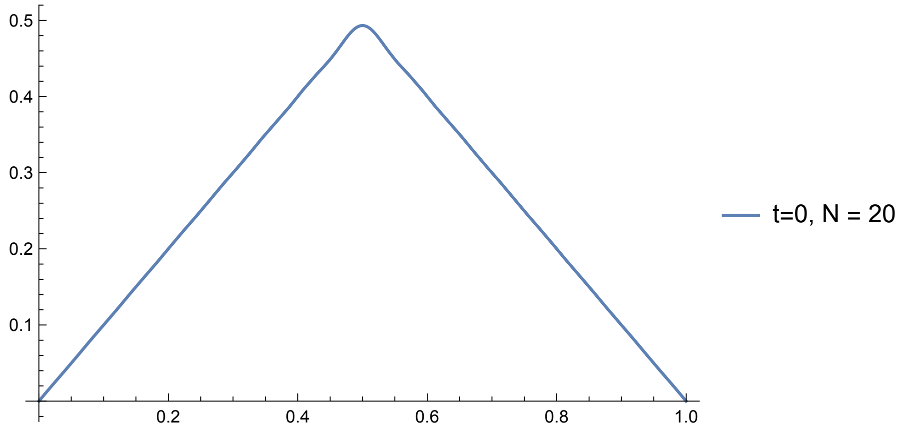
The following plots illustrate how the initial data is represented by the Fourier series. As we take more and more terms, the series reproduces the sharp angle of the initial data./

In[15]:=
```
Plot[{ F4[x, 0.0][10]} , {x, 0, 1}, PlotLegends → {"t=0, N = 10"} ]
Plot[{ F4[x, 0.0][30]} , {x, 0, 1}, PlotLegends → {"t=0, N = 20"} ]
Plot[{ F4[x, 0.0][100]} , {x, 0, 1}, PlotLegends → {"t=0, N = 100"} ]
```

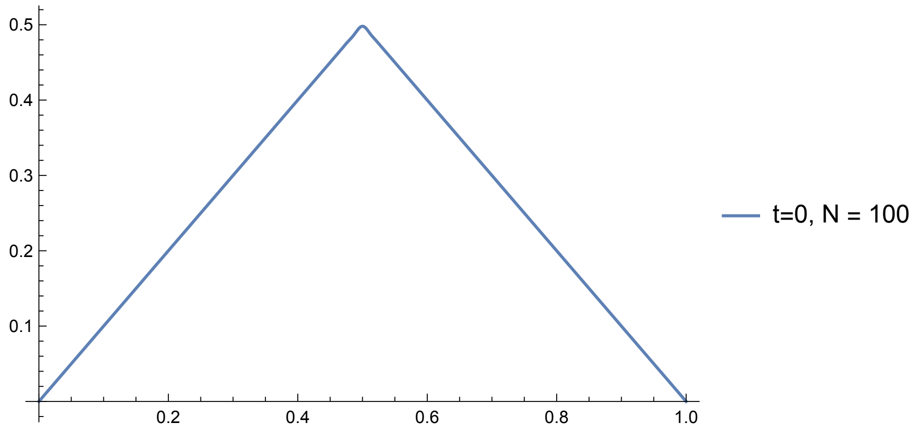
Out[15]=



Out[16]=



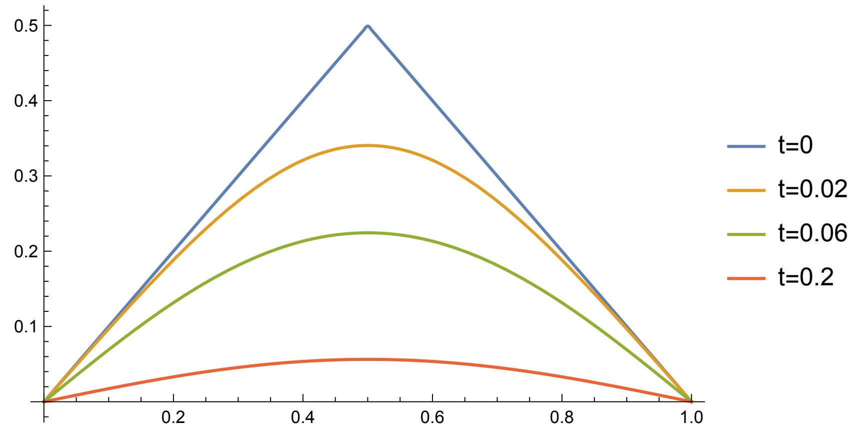
Out[17]=



Now we plot the solution at following times. Notice that the solution becomes smooth as soon as t>0.

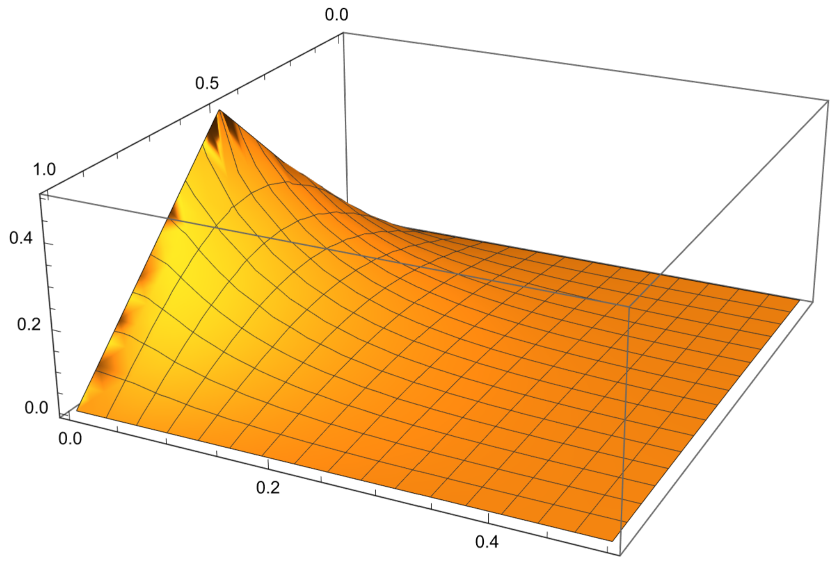
In[18]:=
```
Quiet[
 Plot[{ F4[x, 0.0][200],  F4[x, 0.02][200],  F4[x, 0.06][200],  F4[x, 0.2][200]} ,
  {x, 0, 1}, PlotLegends → {"t=0", "t=0.02", "t=0.06", "t=0.2"} ]]
```

Out[18]=



In[19]:=
```
Plot3D[{F4[x, t][200]}, {x, 0, 1}, {t, 0, 1 / 2}, PlotRange → All]
```

Out[19]=



Problema di Neumann (bordi isolati termicamente)

Problem: a bar of length L = 1, has initial temperature distribution at t=0 given by u(x,0) = 50 x (1-x), x in [0,1]. The two ends of the bar are kept insulated (= Neumann boundary condition). Find the temperature distribution at t>0 . Assume the diffusivity coefficient is $\alpha = 1$.

To solve this problem we use the Fourier method.

In[ ]:=
```
Clear[F5]
```

```
(* Find Fourier coeffcients *)
```

In[22]:=
```
(* The zeroth coefficient *)
d0 = 2 Integrate[ 1 x (1 - x), {x, 0, 1}] 50
```

Out[22]=

$$\frac{50}{3}$$

```
(* the other coefficients *)
2 Integrate[ Cos[n Pi x] x (1 - x), {x, 0, 1}]
```

Out[ ]=

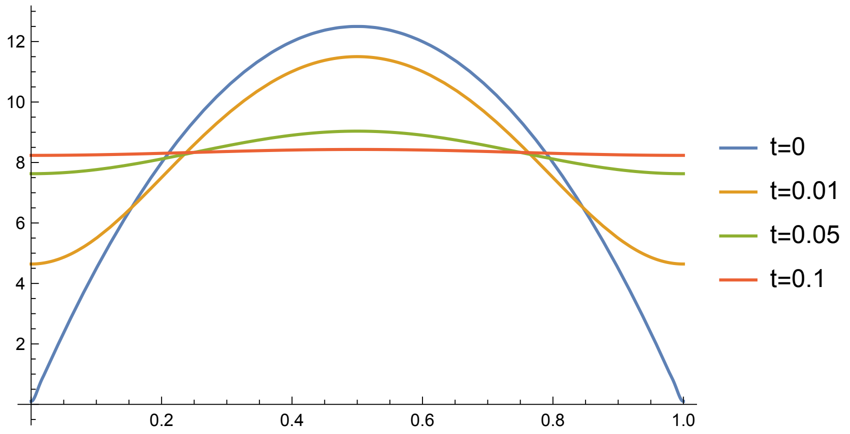
$$-\frac{2\ (n\,\pi + n\,\pi\,\mathrm{Cos}[n\,\pi] - 2\,\mathrm{Sin}[n\,\pi])}{n^3\,\pi^3}$$

In[21]:=
```
F5[x_, t_][NN_] := d0 / 2 +
  Sum[ Cos[n Pi x] E^(- n^2 Pi^2 α t) 50 (-2 / n^2 / Pi^2) (1 + (-1)^n), {n, 1, NN, 1}]
```

Solution at different times. Notice that, despite the fact that the initial data did not satisfy the boundary condition (i.e., u_x was different from zero at the edges, the condition is satisfied as soon as t>0. This is due to the correct choice of eigenfunctions we expand on (the cosines). We can allow for such small violations of the boundary conditions in the initial data. The Fourier series will take care to restore them as t>0.

In[23]:=
```
Quiet[Plot[ {F5[x, 0][100], F5[x, 0.01][100],
    F5[x, 0.05][100], F5[x, 0.1][100]}, {x, 0, 1}, PlotRange → All,
  PlotLegends → {"t=0", "t=0.01", "t=0.05", "t=0.1"}]]
```

Out[23]=



In[ ]:=
```
Plot3D[F5[x, t][100], {x, 0, 1}, {t, 0, 0.2}, PlotRange → All]
```

Out[ ]=