1 Exercises on 2nd order PDEs

Note: In all these exercises, Fourier coefficients can be expressed as integrals. It is not required to compute the integrals explicitly.

1.1 Heat equation

Hint: When the problem has non-homogeneous boundary conditions, it is convenient to subtract an appropriate function in order to get a problem with homogeneous boundary conditions.

Ex. 1 . One-dimensional problem Consider a bar of length L with heat diffusivity coefficient α. The bar's ends are kept at constant temperatures $u(0,t) = T_1$, $u(L,t) = T_2$. The initial temperature is $u(x, t = 0) = 4(L - x)^2 x^2$.

- What is the temperature distribution at $t = \infty$? (This can be answered without explicit calculations).
- Write the solution at generic times.

Ex. 1 b . Consider the same setup as above, with initial condition a constant $u(x, t = 0) = T_0$, and the two ends both kept at constant temperature $T = 0$, i.e. $u(0,t) = u(L,t) = 0$. Write the solution at generic times as a Fourier series.

Ex. 2 - One-dimensional, different boundary conditions A bar of length $L = \pi$ is initially at temperature $T = 0$ at time $t = 0$. At times $t > 0$, the left end of the bar is insulated, while the right end is kept at constant temperature $T = 50$. Find an expression for the time-dependent temperature distribution.

Hint: The fact that the left end is insulated means that there is no heat flow across this end. Remember that the heat flow is proportional to the gradient of the temperature.

Ex. 3 - With time-dependent boundary conditions At time $t = 0$, a bar of length L has uniform temperature $u(x, t = 0) = 0$, $0 \leq x \leq L$. For $t > 0$, the endpoints of the bar are heated such that

$$u(x = 0, t) = 3t, \quad u(x = L, t) = 4t, \tag{1.1}$$

for $0 \leq t \leq 1$. What is the temperature distribution at $t = 1$?

Hint: Do a subtraction in order to get a homogeneous boundary condition. Notice that in this case this will lead you to a non-homogeneous PDE to solve. You can solve it using a Fourier decomposition with time-dependent coefficients (the same trick that we used to solve the problem of the wave equation subject to an external force). Derive the ODE satisfied by the coefficients and solve it.

Ex. 4 - Sphere Write the equations describing the cooling of a sphere of radius R, with initial uniform temperature $u(\vec{x}, t = 0) = T_0 > 0$, which is immersed in a space with uniform temperature $T_{\text{ext}} = 0$.

Hint: use spherical coordinates. See formulas at the end of the file for the relevant eigenfunctions (there are some simplifications given this initial distribution).

1.2 Wave equation

Ex. 5 - 1D A string of length L, fixed at the points $x = 0$ and $x = L$ on the horizontal axis, has initially a displacement of the form $u(x, t = 0) = \sin^2(\frac{x\pi}{L})$, and its transversal velocity is $\frac{\partial}{\partial t} u(x, t)\big|_{t=0} = 0$. The endpoints of the strings are kept fixed. The propagation speed for waves on the string is v.

- Is the motion of the string for $t > 0$ periodic? If yes, what is its period (i.e., minimal time such that the motion repeats itself)?
- Derive an explicit expression for $u(x, t)$ for $t > 0$ (it can be written as an infinite Fourier series, which does not need to be summed. The coefficients should be defined explicitly, as integrals).

Hint: The first point can be answered both using the method of images and using the Fourier decomposition method. Try to deduce the answer using both methods.

Ex. 6 - Vibrations of a drum Consider a circular drum of radius R. The propagation speed for waves on the drum is v.

- What are the vibration frequencies of the drum?
- Suppose that at time $t = 0$ the displacement of the drum's membrane is $u(r, \theta, t = 0) = (r^2 - R^2)\cos\theta$, and its initial velocity is $\partial_t u(r, \theta, t = 0) = 0$. Write an explicit solution using the eigenfunctions method for the displacement of the membrane. The coefficients of the series can be defined as explicit integrals, but there is no need to compute the integrals or sum the series.

Hint: To recall the form of the eigenfunctions in this case, see notes at the end of the file.

Ex. 7 - A different drum Consider a second drum (with the same propagation speed v) which has the shape of a quarter of a disk, namely in radial coordinates, $0 \le r \le R$, $0 \le \theta \le \frac{\pi}{2}$.

- What are the vibration frequencies of the drum?
- Suppose that at time $t = 0$ the displacement of the drum's membrane is $u(r, \theta, t = 0) = 0$, and its initial velocity is $\partial_t u(r, \theta, t = 0) = \sin(2\theta)(r - R) + \sin(4\theta)(r - R)^2$. Write an explicit solution using the eigenfunctions method for the displacement of the membrane.

1.3 Laplace/Poisson equation

Ex. 8 - Dirichlet rectangle A square metal plate of side π ($0 \le x,\ y \le \pi$), has its sides kept at the following temperatures:

$$T(x,0) = T(\pi, y) = 0, \quad T(0,y) = \sin y, \quad T(x,\pi) = 5\sin 2x - 7\sin 8x, \quad 0 \le x,\ y \le \pi. \tag{1.2}$$

What is the equilibrium temperature of the plate?

Ex. 9 - Neumann rectangle Consider the Laplace equation $u_{xx} + u_{yy} = 0$ on a rectangular domain of sides L, M ($0 \le x, \le L$, $0 \le y \le M$), with Neumann boundary conditions

$$u_x(x,0) = u_x(L,y) = u_y(x,M) = 0, \quad u_y(x,0) = f(x). \tag{1.3}$$

Write a form for the solution using Fourier's method. Specify any consistency conditions that are required.

Ex. 10 - Disk Consider Laplace equation $\Delta^{(2D)}u = 0$ in the interior of a disk of radius R, with boundary condition $u(R,\theta) = f(\theta)$ on the boundary of the disk.

- Using the Green's function method discussed in the lecture, write down explicitly the solution at an interior point $u(r,\phi)$, as a function of the boundary data.
- Derive an alternative form of the solution using the Fourier method.

Ex. 11 - Annulus Consider Laplace equation in the interior of an annulus region, namely the region defined by $R_1 < r < R_2$, $r = \sqrt{x^2+y^2}$.

Consider the case $R_2 = 2$, $R_1 = 1$. Using a decomposition into eigenfunctions in radial coordinates, solve the following boundary value problem:

$$\Delta^{(2D)}u = 0, \quad R_1 < r < R_2, \quad u(r_2,\theta) = 1 + 2\cos\theta + \cos 2\theta, \quad u(r_1,\theta) = \sin 2\theta. \tag{1.4}$$

1.4 Relevant formulas

1.4.1 Fourier orthogonality

The following orthogonality properties of the sine/cosine functions are the basis of Fourier analysis:

$$\frac{1}{L}\int_0^{2L} \sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{2L}) = \delta_{mn}, \quad n,m \in \mathbb{N}^+ \tag{1.5}$$

$$\frac{1}{L}\int_0^{2L} \cos(\frac{n\pi x}{L})\cos(\frac{m\pi x}{2L}) = \frac{\delta_{mn}}{2^{\delta_{m,0}}}, \quad n,m \in \mathbb{N}, \tag{1.6}$$

$$\frac{1}{L}\int_0^{2L} \cos(\frac{n\pi x}{L})\sin(\frac{m\pi x}{2L}) = 0. \tag{1.7}$$

In some types of applications we often need the following relations:

$$\frac{2}{L}\int_0^L \sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{2L}) = \delta_{mn}, \quad n,m\in\mathbb{N}^+, \tag{1.8}$$

$$\frac{2}{L}\int_0^L \cos(\frac{n\pi x}{L})\cos(\frac{m\pi x}{2L}) = \frac{\delta_{mn}}{2^{\delta_{m,0}}} \quad n,m\in\mathbb{N},, \tag{1.9}$$

which are a simple consequence of the two above (notice that instead (1.7) above does not generalize to the half-interval).

In problems with mixed Dirichlet-Neumann boundary conditions, the following are useful:

$$\frac{2}{L}\int_0^L \sin(\frac{(n+\frac{1}{2})\pi x}{L})\sin(\frac{(n+\frac{1}{2})\pi x}{2L}) = \delta_{mn}, \quad n,m\in\mathbb{N}, \tag{1.10}$$

$$\frac{2}{L}\int_0^L \cos(\frac{(n+\frac{1}{2})\pi x}{L})\cos(\frac{(n+\frac{1}{2})\pi x}{2L}) = \delta_{mn}, \quad n,m\in\mathbb{N}, \tag{1.11}$$

The standard Fourier decomposition of a function defined on an interval $[0,2L]$ involves both sine and cosine functions.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(n\pi x/L) + b_n\sin(n\pi x/L)\right). \tag{1.12}$$

this defines a function of period $2L$. The coefficients can be found explicitly using orthogonality. For instance, integrating $\sin(\frac{n\pi x}{L})$ against the function, and using the orthogonality above, we deduce $b_n = \frac{1}{L}\int_0^{2L} dx\ \sin(\frac{n\pi x}{L})f(x)$.

Often in PDE problems we want to use a different type of series expansion, which is needed to capture the boundary conditions. Typically, we use a basis of trigonometric functions which corresponds to the standard Fourier decomposition of a *larger* interval.

For instance, consider a function $f(x)$ defined on the interval $x\in[0,L]$. In problems where we want to impose Dirichlet-Dirichlet boundary conditions, we use an expansion as a *sine-series*:

$$f(x) = \sum_{n=1}^{\infty} d_n\sin(n\pi x/L). \tag{1.13}$$

Notice that this defines a function of period $2L$, which is odd: $f(x) = -f(-x)$. It is the *odd extension* of the original function. The coefficients are given by $d_n = \frac{2}{L}\int_0^L dx\ \sin(\frac{n\pi x}{L})f(x)$, as can be deduced using the orthogonality relations (1.8) above.

1.4.2 Radial coordinates

Radial coordinates in 2D The Laplace operator in $2D$ is $\Delta^{(2D)} \equiv \partial_x^2 + \partial_y^2$. In radial coordinates it becomes:

$$\partial_r^2 + \frac{\partial_r}{r} + \frac{\partial_\theta^2}{r^2}, \tag{1.14}$$

where the radial coordinates are $r=\sqrt{x^2+y^2}$, $\theta = \arccos\frac{x}{r} = \arcsin\frac{y}{r}$.

Notable radial equation in 2D - Laplace The ODE

$$R''(r) + \frac{R'(r)}{r} - \frac{m^2 R(r)}{r^2} = 0 \tag{1.15}$$

arises studying Laplace equation $\Delta u = 0$ in radial coordinates in 2D.

This equation has two independent solutions: $R(r) = r^{\pm m}$ for $m > 0$, and for $m = 0$ the two independent solutions are: $R(r) = \text{const}$, $R(r) = \log r$.

Notable radial equation in 2D - Helmholtz The ODE

$$R''(r) + \frac{R'(r)}{r} + \frac{\lambda\, r^2\, R(r) - m^2 R(r)}{r^2} = 0 \tag{1.16}$$

arises studying Helmholtz equation $\Delta u = -\lambda u$ in radial coordinates in 2D.

It can be transformed into Bessel equation for $y(x) = R(\sqrt{\lambda} x)$: $y''(x) + \frac{y'(x)}{x} + \frac{x^2 - m^2}{x^2} y(x) = 0$.

Assume $\mathrm{Re}(m) \geq 0$. Then the solution of Bessel equation with behaviour x^m for $x \sim 1$ is called *Bessel function of the first kind* $J_m(x)$. It has infinitely many zeros on the positive real axis (including $x = 0$ if $m > 0$).

Bessel functions satisfy the following orthogonality relations:

$$\int_0^1 x J_\alpha(\mu_{\alpha,k} x) J_\alpha(\mu_{\alpha,l} x) \propto \delta_{kl}, \tag{1.17}$$

$\forall \alpha$, $k = 1, 2, \ldots$, where $\mu_{\alpha,k}$ denote the zeros, $J_\alpha(\mu_{\alpha,k}) = 0$, $k = 1, 2, \ldots$.

Radial coordinates in 3D The Laplace operator in $3D$ is $\Delta^{(3D)} \equiv \partial_x^2 + \partial_y^2 + \partial_z^2$. In radial coordinates it becomes:

$$\partial_r^2 + \frac{2\partial_r}{r} + \frac{\partial_\theta^2 + \cot\theta \partial_\theta + \frac{\partial_\phi^2}{\sin^2\theta}}{r^2}, \tag{1.18}$$

where the radial coordinates are $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \arccos \frac{z}{r}$, $\phi \arcsin \frac{y}{r \sin\theta} = \arccos \frac{x}{r \sin\theta}$.

Notable radial equation in 3D - Laplace The ODE

$$R''(r) + \frac{2R'(r)}{r} - \frac{l(l+1)R(r)}{r^2} = 0, \tag{1.19}$$

arises studying Laplace equation $\Delta u = 0$ in radial coordinates in 3D.

This equation has two independent solutions: $R(r) = r^l$ and r^{-l-1}.

Notable radial equation in 3D - Helmholtz Studying the more general Helmholtz equation, $\Delta^{(3D)} u = -\lambda u$, the radial part leads to the ODE

$$R''(r) + \frac{2R'(r)}{r} - \frac{l(l+1)R(r) - \lambda r^2 R(r)}{r^2} = 0, \tag{1.20}$$

This equation can also be mapped to Bessel equation doing the substitution $y(x) = R(\sqrt{\lambda}x)/\sqrt{x}$. **check it!**

Assume $\mathrm{Re}(l) \geq 0$. The solution with behaviour $R(r) \sim r^l$ can be written as $R(r) \propto \frac{J_{l+\frac{1}{2}}(\sqrt{\lambda}r)}{\sqrt{\lambda\, r}}$.

Note: The simplest case is for $l = 0$ (which is relevant to decompose the angle-independent part of solutions of Helmholtz's equation on the sphere). In this case, the eigenfunction is simple because $J_{\frac{1}{2}}(x)/\sqrt{x} = \sqrt{\frac{2}{\pi}} \frac{\sin x}{x}$.