"INVISCID" BURGERS EQUATION AND THE GRADIENT CATASTROPHE

LET US USE THE METHOD OF CHARACTERISTICS TO STUDY A PARTICULARLI INTERESTING PDE, WHICH CAPTURES SOME IMPORTANT PHENOMENA IN FLUID DYNAMICS: THE "INVISCID" BURGERS EQUATION

$$\partial_t u(x,t) = u(x,t)\, \partial_x u(x,t)$$

"INVISCID" MEANS WITHOUT VISCOSITY. LATER WE WILL STUDY THE FULL BURGERS' EQUATION:

$$\partial_t u = u\, u_x + \underbrace{\mu\, u_{xx}}_{\text{VISCOSITY TERM}} \qquad \mu > 0$$

MEANING IN FLUID DYNAMICS (SKETCH):

IN FLUID DYNAMICS THE MAIN VARIABLES ARE

$\rho(\vec{x},t)$: DENSITY OF THE FLUID

$u(\vec{x},t)$: VELOCITY OF THE FLUID AT POINT $\vec{x}$, TIME t

$\vec{x} = x_1, x_2, \dots x_d$

THEY SATISFY RELATIONS EXPRESSING CONSERVATION OF MASS, MOMENTUM AND ENERGY.

IN PARTICULAR THE LOCAL CONS. OF MASS IS EXPRESSED AS:

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{u} \cdot \vec{\nabla} \rho + \rho (\vec{\nabla} \cdot \vec{u}) = 0}$$

(FIRST EULER EQUATION)

THIS EQUATION TELLS US THAT MASS CAN BE TRANSPORTED ALONG THE FLOW OF THE FLUID WITH VELOCITY $\vec{u}$, BUT IS NOT CREATED OR DESTROYED.

LOCAL CONS. OF MOMENTUM GIVES:

$$\boxed{\frac{\partial(\rho u_i)}{\partial t} + \vec{u} \cdot \vec{\nabla}(\rho u_i) + \rho u_i (\vec{\nabla} \cdot \vec{u}) = -\partial_i P(\vec{x},t) + F_i(\vec{x},t) \qquad (i = 1, \dots, d)}$$

(2nd EULER EQUATION – (REALLY A SET OF d EQUATIONS))

ABOVE, $\vec{F}(\bar{x},t)$ IS THE EXTERNAL FORCE (PER UNIT VOLUME) ACTING ON THE FLUID AT $(\vec{x},t)$, WHILE P = PRESSURE (SO $-\vec{\nabla} P$ = FORCE DUE TO PRESSURE PER UNIT VOLUME).

IF WE LOOK AT THE ONE-DIMENSIONAL CASE, $(d=1)$, IN THE ABSENCE OF FORCES $(F(x,t)=0)$, AND WITHOUT PRESSURE $(P=0)$, WE GET:

- $\frac{\partial}{\partial t}\rho + u\,\partial_x \rho + \rho\,\partial_x u = 0$
- $u\underbrace{\left(\frac{\partial}{\partial t}\rho + u\,\partial_x\rho + \rho\,\partial_x u\right)}_{=0} + \rho\,\partial_t u + \rho\, u\, u_x = 0$

AND USING THE FIRST EQ. IN THE SECOND, WE GET

$$\boxed{u_t + u\,u_x = 0}$$

INVISCID B. EQUATION

FOR THE VELOCITY OF THE FLUID.

NOTE: THE NAVIER-STOKES EQUATIONS ADD ONE INGREDIENT TO THE EULER EQUATIONS: VISCOSITY, WHICH REPRESENTS THE FRICTION

BETWEEN PARTS OF THE FLUID MOVING AT DIFFERENT VELOCITIES: $\vec{F}_{viscosity} = \mu \cdot \Delta \vec{u}$,

COEFFICIENT

WITH $\Delta = \vec{\nabla} \cdot \vec{\nabla} = \sum_{i=1}^{d} \left(\frac{\partial}{\partial x_i}\right)^2$.

SO THE 2nd EULER EQ. BECOMES

NAVIER-STOKES EQ:

$$\frac{\partial}{\partial t}(\rho u_i) + \vec{u} \cdot \vec{\nabla}(\rho u_i) + \rho u_i \vec{\nabla} \cdot \vec{u}$$

$$= F_i^{ext}(\vec{x},t) - \partial_i P(\vec{x},t)$$

$$+ \mu \Delta(u_i).$$

REPEATING THE DERIVATION ABOVE FOR $d=1$, $F^{ext} = P = 0$, WE GET THE FULL

BURGERS EQUATION:

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u = \mu \partial_x^2 u$$

SOLVING BURGERS EQ. ($\mu = 0$) WITH THE METHOD OF CHARACTERISTICS

LET US CONSIDER THE SIMPLEST CASE:

$$u_t + u u_x = 0$$

WE WOULD LIKE TO UNDERSTAND THE EVOLUTION OF AN INITIAL CONDITION

$$u(x, t=0) = u_0(x).$$

THE EQUATION OF CHARACTERISTIC CURVES ARE:

(ℓ = PARAMETER ALONG THE CURVE)

$$\begin{cases} \dfrac{dt}{d\ell} = 1 \\[2ex] \dfrac{dx}{d\ell} = u \\[2ex] \dfrac{du}{d\ell} = 0 \end{cases}$$

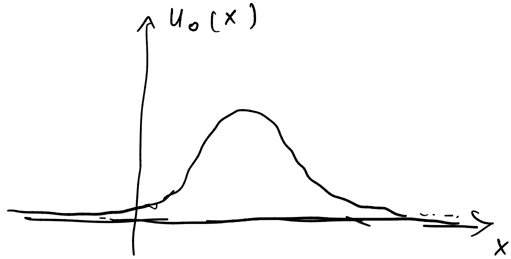
THE FIRST EQ. TELLS US THAT ℓ CAN BE IDENTIFIED WITH THE TIME VARIABLE t.

THE OTHER TWO EQS. TELL US THAT THE CHARACTERISTIC CURVES ARE STRAIGHT LINES, WITH SLOPE EQUAL TO THE VALUE OF u. u REMAINS CONSTANT ALONG EACH LINE.

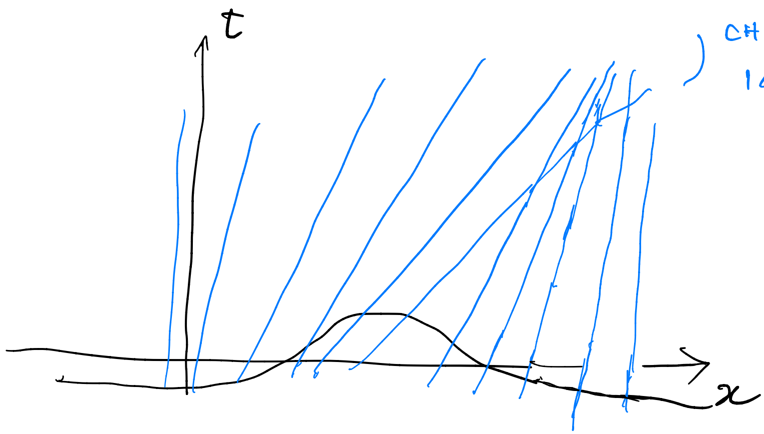
INTUITION

PICTURE:

TAKE INITIAL CONDITION:



↳ IT DEFINES THE SLOPE OF CHAR. CURVES:



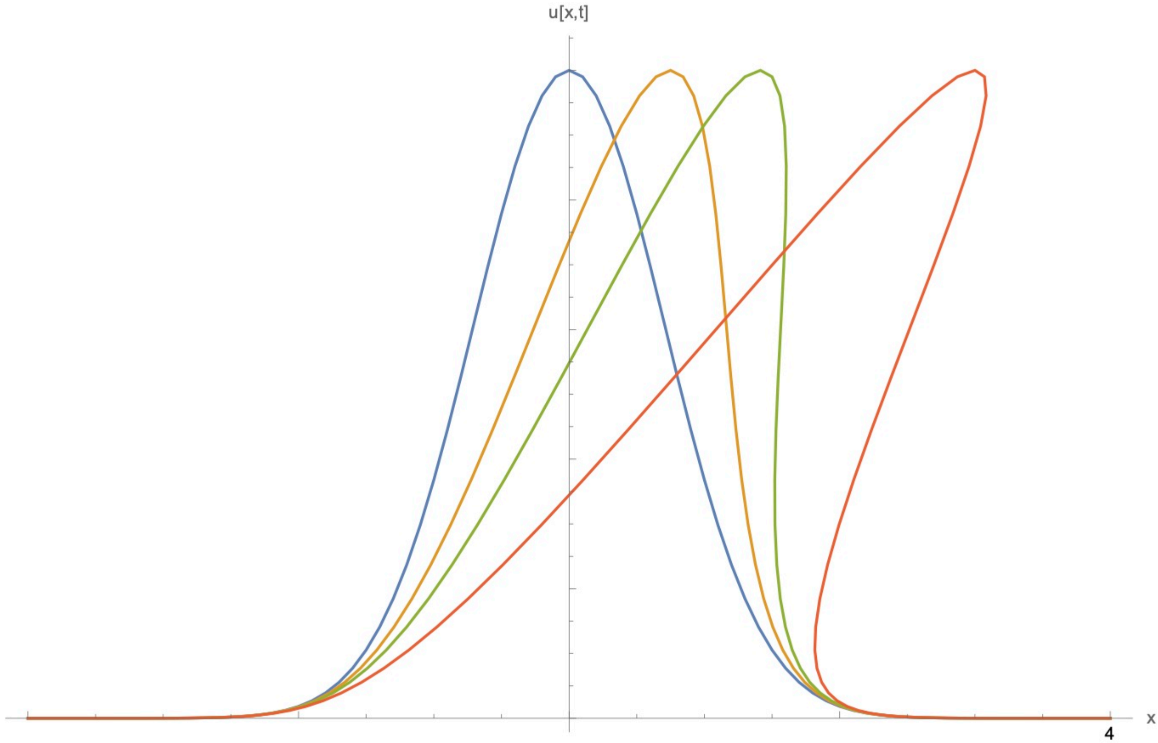
u IS CONSTANT ALONG EACH CURVE

→ BECAUSE THE CURVES HAVE SLOPE DEPENDING ON VALUES OF u, IF u_0 IS NOT CONSTANT THEY MAY HAVE REGIONS WHERE THEY TEND TO FOCUS AND CAN INTERSECT.

CONTRAST THIS WITH THE STANDARD TRANSPORT EQUATION, $u_t + b\,u_x = 0$, WHERE THE CHARACTERISTICS WERE ALL STRAIGHT LINES WITH SLOPE b.

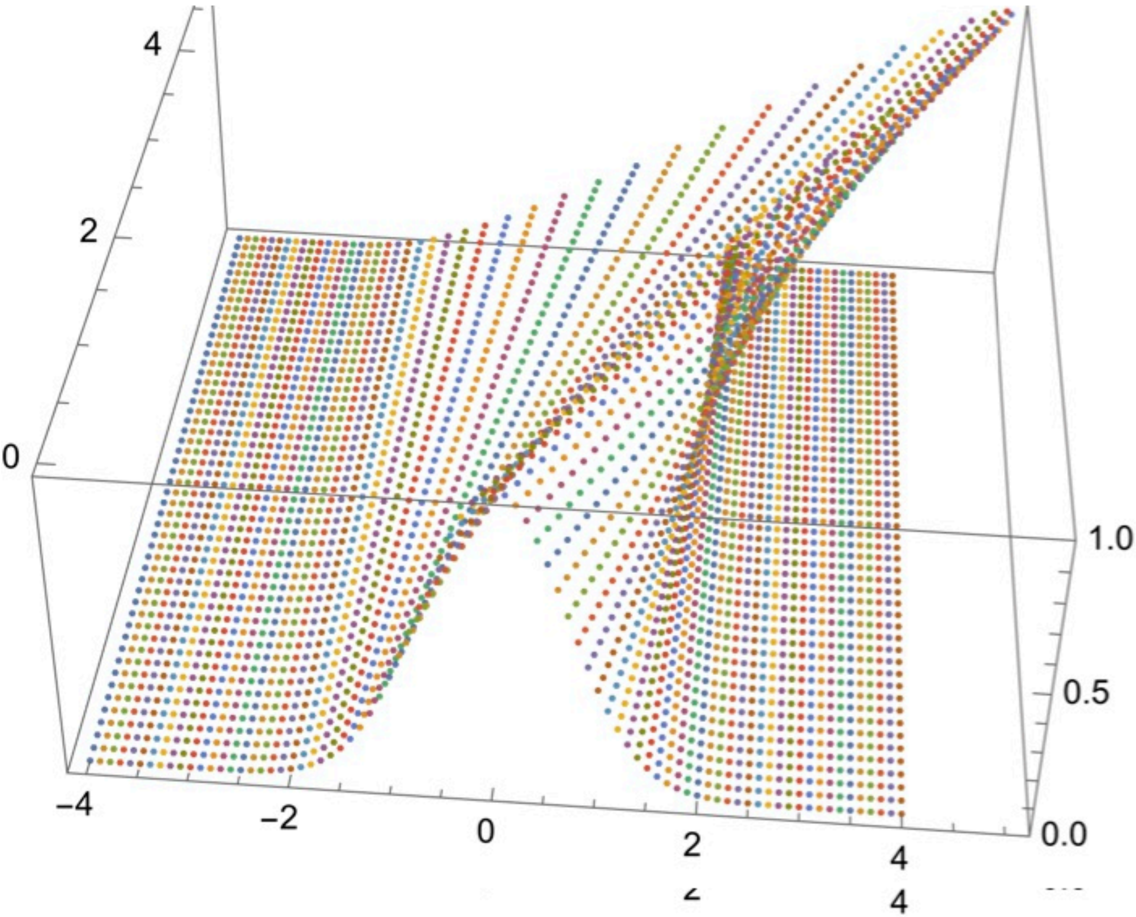
WHAT HAPPENS TO $u(x,t)$ AS THE CHARACTERISTICS FOCUS AND THEN INTERSECT?

ILLUSTRATION FOR INITIAL PROFILE $u_0(s) = e^{-s^2}$.



Solution at times
t = 0 (blue) , t = 0.75 (yellow) , t = t_c = Sqrt[2] (green) , t = 3 > t_c .
Notice the phenomenon of shock formation (= infinite space derivative)
at t = t_c

THERE IS A CRITICAL TIME WHEN THE SOLUTION DEVELOPS AN INFINITE GRADIENT $|u_x(x,t)| \to \infty$ FOR $t \to t_c^-$, EVEN IF THE INITIAL DATA ARE SMOOTH.



3D plot (x,t,u) of the solution of the Burgers equation with initial condition u(x,0) = e^(-x^2).
Notice the solution becomes multi-valued as the characteristic curves cross.
The plot was constructed by using the characteristic method.
On Moodle you can see the simple commands to generate the plot in Mathematica

THIS PHENOMENON IS CALLED "WAVE BREAKING" OR "SHOCK FORMATION" OR "GRADIENT CATASTROPHE", OR "BLOW UP".

AFTER t_c, THE CHARACTERISTICS CROSS, AND THE METHOD OF CHARACTERISTICS PRODUCES A MULTI-VALUED SOLUTION.

TO UNDERSTAND WHAT HAPPENS QUANTITATIVELY, LET US SOLVE THE CHARACTERISTICS EQUATIONS WITH THE PROPER INITIAL CONDITIONS.

INTRODUCING: ℓ : PARAMETER ALONG THE CHARACT. CURVE

s : PARAMETRISES THE INITIAL CURVE

$(\vec{\gamma}(s) = (s, 0))$

$$\mathcal{U}(s,\ell) \equiv u(x(s,\ell), t(x,\ell)),$$

WE HAVE:

$$\begin{cases} \dfrac{\partial}{\partial \ell} t(s,\ell) = 1, & t(s,0) = 0. \\ \dfrac{\partial}{\partial \ell} x(s,\ell) = \mathcal{U}(s,\ell), & x(s,0) = s. \\ \dfrac{\partial}{\partial \ell} \mathcal{U}(s,\ell) = 0, & \mathcal{U}(s,0) = u_0(s). \end{cases}$$

THE SOLUTION IS :

$$\boxed{\begin{aligned} t(s,\ell) &= \ell . \\ x(s,\ell) &= s + u_0(s)\,\ell \\ U(s,\ell) &= u_0(s) \end{aligned}} \longrightarrow \begin{aligned} &u(x,t) \\ &= U(s,t) \\ &= u_0(s). \end{aligned}$$

TO EXPRESS THE SOL. IN THE ORIGINAL VARIABLES WE NEED TO INVERT THE MAP

$$(s,\ell) \rightarrow (x,t) .$$

IN GENERAL THIS CANNOT BE DONE EXPLICITLY. BUT WE CAN WRITE A GENERAL IMPLICIT FORMULA FOR THE SOLUTION.

SINCE :

$$\ell = t$$

$$s = x - u_0(s)\,t = x - u(x,t)\,t ,$$

$$\boxed{u(x,t) = u_0\big(x - u(x,t)\cdot t\big)}$$

(IMPLICIT FORMULA).

WE CAN USE THIS FORMULA TO COMPUTE THE TIME t_c WHEN $|u_x| \to \infty$.

IN FACT, DIFFERENTIATING THE FORMULA ABOVE, WE FIND:

$$u_x(x,t) = u_0'(x-u(x,t)\,t) \cdot \partial_x\left(x - u(x,t)\,t\right)$$

$$= u_0'(x-u(x,t)\,t) \cdot \left(1 - t\,u_x(x,t)\right)$$

$$\rightarrow \boxed{u_x = \frac{u_0'(x-u(x,t)\,t)}{1 + t\,u_0'(x-u(x,t)\,t)}}$$

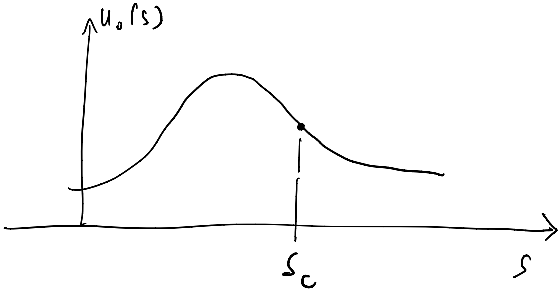
THE SINGULARITY IS DUE TO

$$\boxed{1 + t\,u_0'(s) \to 0} \qquad (\text{FOR } s = x - u(x,t)\,t)$$

THE EARLIEST TIME WHEN THIS CONDITION IS SATISFIED (FOR SOME s) IS:

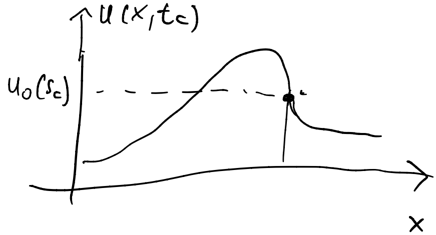
$$\boxed{t_c = \min_s \left(-\frac{1}{u_0'(s)}\right)}, \quad \text{PROVIDED } t_c > 0.$$

THE SINGULARITY IS ASSOCIATED TO THE POINT s_c SUCH THAT $-u_0'(s_c)$ IS MAXIMAL AND > 0.



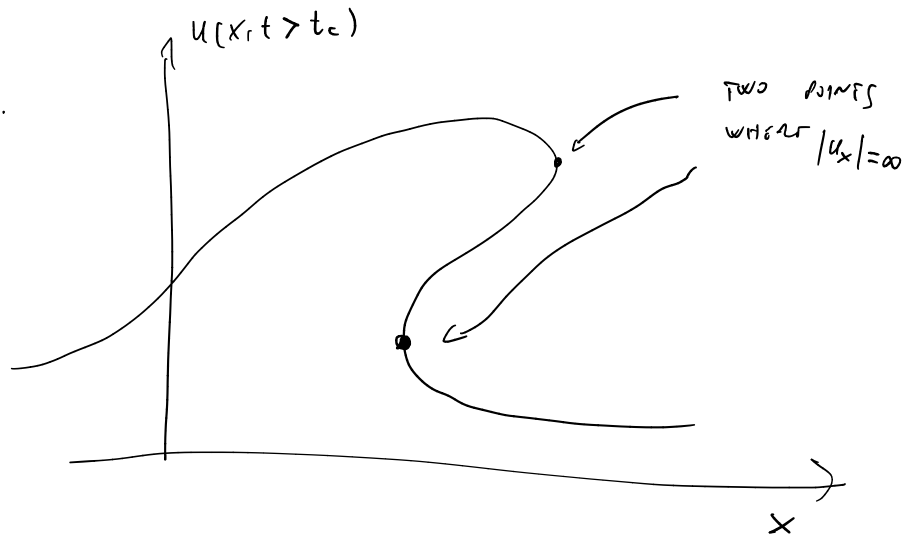
$\longrightarrow \quad u = u_0(s_c)$

AT THE BREAKING POINT AT $t = t_c$



FOR $t > t_c$, THE CONDITION $1 + t\, u_0'(s)$ ADMITS MORE THAN ONE SOLUTION.

e.g.

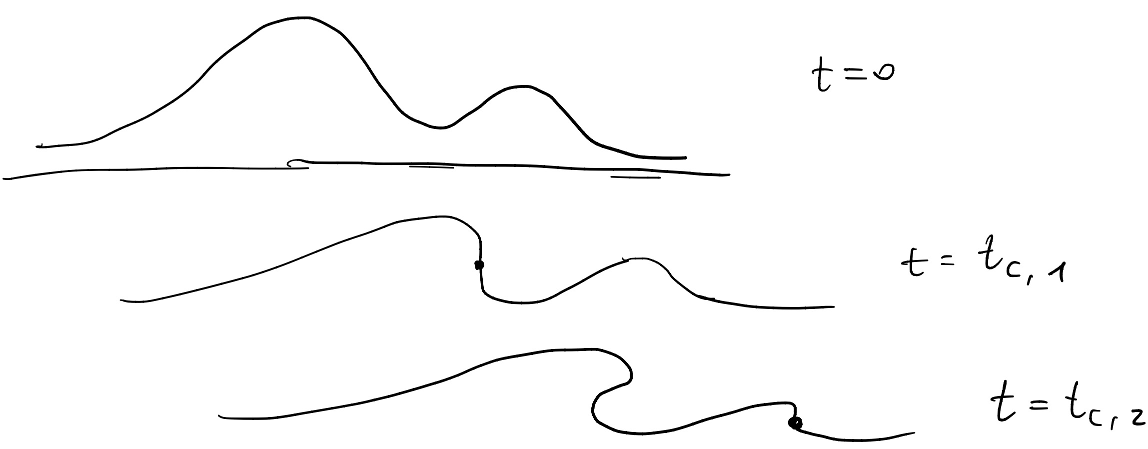


• NOTE THAT THE VALUE OF t_c DEPENDS ON THE INITIAL PROFILE.

FOR SOME INITIAL PROFILES THERE MAY BE NO SINGULARITY (THIS HAPPENS IF $u_0'(s) > 0$ FOR ALL s).

THE SINGULARITY ALWAYS HAPPENS WHEN THERE IS A LOCAL POSITIVE MAXIMUM FOR $u_0'(s)$.

IF THE INITIAL PROFILE HAS SEVERAL ISOLATED MAXIMA FOR $u_0'(s)$, THEN THERE IS A SEQUENCE OF SHOCKS FORMING AT DIFFERENT TIMES.



* NOTE THAT THE CONDITION

$$1 + t\, u_0'(s) = 0$$

(WHICH WE DERIVED AS THE CONDITION THAT

$$u_x(x,t) \to \infty\ )$$

ALSO HAS ANOTHER NATURAL INTERPRETATION:

IT MEANS THAT THE MAP BETWEEN

$(s, \ell) \longrightarrow (x, t)$ IS NOT INVERTIBLE

IN FACT THE MAP IS

$$t = \ell$$
$$x = s + u_0(s)\, \ell$$

ITS JACOBIAN IS

$$J = \begin{vmatrix} \frac{\partial t}{\partial \ell} & \frac{\partial t}{\partial s} \\ \frac{\partial x}{\partial \ell} & \frac{\partial x}{\partial s} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ u_0(s) & 1 + u_0'(s)\ell \end{vmatrix} = 1 + \ell u_0'(s)$$

(WITH $\ell = t$)

POINTS WHERE $1 + t\, u_0'(s) = 0$ ARE POINTS WHERE

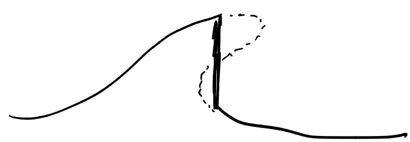
$J = 0 \Longleftrightarrow$ THE MAP IS NOT INVERTIBLE.

COMMENT

IN MANY APPLICATIONS (e.g. FLUIDS) THE FACT THAT THE SOLUTION BECOMES MULTIVALUED AFTER t_c, MEANS THAT OUR MODEL BREAKS DOWN, FOR EXAMPLE PHYSICALLY THE VELOCITY OF THE FLUID CANNOT BE MULTIVALUED.

WE WILL SEE LATER HOW WE CAN STILL MAKE SENSE OF THE MODEL IN THIS SITUATION. TWO MAIN OPTIONS:

- CONSIDER DISCONTINUOUS SOLUTIONS FOR $t > t_c$ (SHOCK WAVES)



(WE NEED TO DISCUSS HOW TO "CHOP" THEM

- INTRODUCE VISCOSITY WHICH MAKES IT SMOOTH

AND APPROXIMATES THE SHOCK WAVES WITHOUT MULTI-VALUED SOLUTIONS