METHOD OF CHARACTERISTICS FOR FULLY NONLINEAR 1st ORDER PDE's

CONSIDER A GENERIC NONLINEAR EQUATION

$$F(u, u_{x_i}, x_i) = 0, \quad \text{WHERE } i = 1, \dots, n. \quad (n \geq 2)$$

FOR GENERIC $F : \mathbb{R}^{2n+1} \to \mathbb{R}$, THE PDE IS NOT NECESSARILY QUASI-LINEAR.

HOWEVER ONE CAN STILL APPLY THE METHOD OF CHARACTERISTICS!

RECALL THAT IN PREVIOUS CASES:

- FOR A LINEAR PDE IN n VARIABLES, CHARACTERISTIC CURVES OBEYED A SYSTEM OF n ODE's (THEY WERE INDEPENDENT ON u)
- FOR A QUASI-LINEAR PDE, THE CHARACTERISTICS OBEY A SYSTEM OF $n+1$ ODE's (FOR x_i, $i = 1, \dots, n$, AND u ITSELF)

* IN THE GENERAL NONLINEAR CASE, CHARACTERISTIC CURVES OBEY A SYSTEM OF $2n+1$ ODE's.

(ODE's FOR: x_i, u, AND $\partial_{x_i} u$)

NOTATION: WE WILL CALL $u_{x_i} \equiv p_i$.

THE DERIVATION OF THIS SYSTEM OF ODE's IS SOMEWHAT TECHNICAL (IT CAN BE FOUND IN THE BOOK BY STAVROULAKIS & TERSIAN).

LET US TAKE A SHORTCUT, ASSUMING WITHOUT PROOF PART OF THE RESULT.

WE ASSUME
$$\frac{dx_i}{d\ell} = \frac{\partial F}{\partial p_i}$$

(ℓ = PARAMETER ALONG THE CHARACTERISTIC CURVE).

NOW WE NEED TO FIND $\frac{dp_i}{d\ell}$ AND $\frac{du}{d\ell}$

WE HAVE:

$$\frac{du}{d\ell} = \frac{d}{d\ell}\left(u(\vec{x}(\ell))\right) = \sum_{i=1}^{n} \dot{x}_i(\ell)\ \partial_{x_i} u(\vec{x}(\ell))$$

$$= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \cdot p_i$$

AND:

$$\frac{d}{d\ell} p_i = \frac{d}{d\ell}\left(u_{x_i}(\vec{x}(\ell))\right)$$

$$= \sum_{j=1}^{n} \dot{x}_j\ u_{x_i x_j}(\vec{x}(\ell))$$

$$= \sum_{j=1}^{n} \frac{\partial F}{\partial p_j} \cdot u_{x_i x_j}$$

THE LATTER EQUATION IS NOT IN A USABLE FORM YET: WE NEED TO REWRITE IT SO THAT ONLY u, x_i, p_i APPEAR.

TO DO THAT, WE DIFFERENTIATE THE

PDE : $\partial_{x_i} \left(F(u, \vec{x}, \vec{\nabla} u) \right) = 0$

$$\rightarrow 0 = u_{x_i} \cdot \frac{\partial F}{\partial u} + \frac{\partial F}{\partial x_i} + \sum_{j=1}^{n} u_{x_i x_j} \frac{\partial F}{\partial p_j}$$

WE USE THIS IDENTITY TO REWRITE $\frac{d}{dl} p_i$ COMPUTED ABOVE AS :

$$\boxed{\frac{dp_i}{dl} = -p_i \frac{\partial F}{\partial u} - \frac{\partial F}{\partial x_i}}$$

IN CONCLUSION, THE SYSTEM OF CHARACTERISTICS ODE'S FOR THE NONLINEAR PDE IS :

$$\boxed{\begin{cases} \dfrac{dx_i}{dl} = \dfrac{\partial F}{\partial p_i} \\[2ex] \dfrac{dp_i}{dl} = -p_i \dfrac{\partial F}{\partial u} - \dfrac{\partial F}{\partial x_i} \\[2ex] \dfrac{du}{dl} = \displaystyle\sum_{i=1}^{n} p_i \dfrac{\partial F}{\partial p_i} \end{cases}}$$

EXAMPLE | LET US USE THE METHOD ABOVE TO SOLVE THE EIKONAL EQUATION

$$u_x^2 + u_y^2 = 1,$$

WITH INITIAL CONDITION :

$$\begin{cases} u = 1 \quad \text{FOR} \quad x^2 + y^2 = 1 \end{cases}$$

LET US PARAMETRIZE THE "CAUCHY CURVE" (i.e., THE CURVE WHERE THE INITIAL CONDITION IS GIVEN) AS $\vec{\gamma}(s) = (\cos s, \sin s)$.

THEN $u(\vec{\gamma}(s)) = u_0(s) = 1$.

THE CHARACTERISTIC CURVES OBEY : $\left(\begin{array}{l} p_x \equiv u_x \\ p_y \equiv u_y \end{array} \right)$

$$\begin{cases} \frac{\partial}{\partial \ell} x = 2 p_x \\ \frac{\partial}{\partial \ell} y = 2 p_y \\ \frac{\partial}{\partial \ell} u = 2 (p_x^2 + p_y^2) = 2 \\ \frac{\partial}{\partial \ell} p_x = 0 \\ \frac{\partial}{\partial \ell} p_y = 0 \end{cases}$$

WE USE THE NOTATION e.g. $x_i(\ell, s)$

ℓ: PARAMETER ALONG THE CHAR. CURVE

s: PARAMETRISES INITIAL CONDITIONS

WITH INITIAL CONDITIONS:

$$x(\ell=0, s) = \cos s$$

$$y(\ell=0, s) = \sin s$$

$$u(\ell=0, s) = u_0(s) = 1$$

$$p_x(\ell=0, s) = p_{x,0}(s)$$

$$p_y(\ell=0, s) = p_{y,0}(s)$$

SOLUTION:

$$u(\ell, s) = 2\ell + u_0(s)$$

$$p_x(\ell, s) = p_{x,0}(s)$$

$$p_y(\ell, s) = p_{y,0}(s)$$

$$x(\ell, s) = 2 p_{x,0}(s)\,\ell + \cos s$$

$$y(\ell, s) = 2 p_{y,0}(s)\,\ell + \sin s$$

WE NEED TO COMPUTE $p_{i,0}(s)$, i.e., p_i ON THE INITIAL CURVE.

TO DO THAT WE USE THE CONDITIONS:

(1) $$\frac{d}{ds} u_0(s) = \frac{d}{ds} u(\vec{\gamma}(s)) = \sum_{i=1}^{n} \dot{\gamma}_i(s) \cdot p_{i,0}(s)$$

TOGETHER WITH THE PDE ON THE CURVE:

(2) $$F\left(u_0(s), \gamma_i(s), p_{i,0}(s)\right) = 0$$

↳ IN OUR CASE WE GET

$$(1) \quad \frac{d}{ds} u_0(s) = 0 = -p_{x,0}(s) \sin s + p_{y,0}(s) \cos s$$

AND

$$(2) \quad p_{x,0}^2(s) + p_{y,0}^2(s) = 1$$

THIS SYSTEM HAS TWO SOLUTIONS:

$$\begin{cases} p_{x,0}(s) = \cos s \\ p_{y,0}(s) = \sin s \end{cases}$$

OR

$$\begin{cases} p_{x,0}(s) = -\cos s \\ p_{y,0}(s) = -\sin s \end{cases}$$

EACH OF THESE CHOICES LEADS TO A LEGITIMATE SOLUTION TO THE EIKONAL EQ.

LET US PROCEED TAKING THE FIRST CHOICE.

SO, WITH $p_{x,0}(s) = \cos s$, $p_{y,0}(s) = \sin s$,

THE CHARACTERISTIC CURVES ARE:

WITH: $\mathcal{U}(\ell,s) = u(x(\ell,s), y(\ell,s))$

$$\mathcal{U}(\ell,s) = 2\ell + 1$$

$$x(\ell,s) = (2\ell+1)\cos s$$

$$y(\ell,s) = (2\ell+1)\sin s$$

$$p_x(\ell,s) = \cos s$$

$$p_y(\ell,s) = \sin s$$

WE CAN EASILY ELIMINATE ℓ, s TO FIND:

$$\boxed{u(x,y) = \sqrt{x^2 + y^2}}$$

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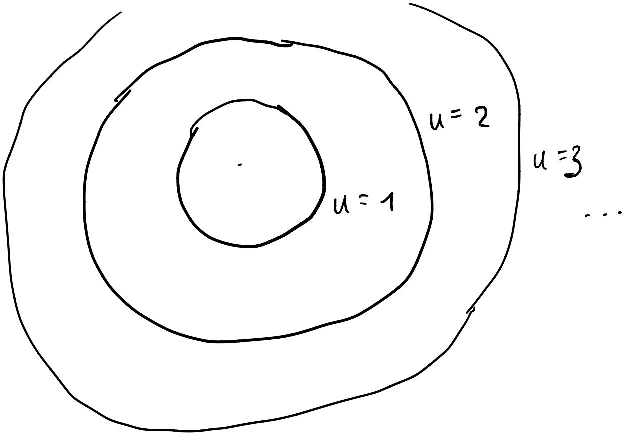
* EXERCISE: CHECK THAT BY TAKING THE SECOND CHOICE FOR $p_{x,0}(s)$, $p_{y,0}(s)$, ONE GETS:

$$u(x,y) = 2 - \sqrt{x^2 + y^2}.$$

ASIDE: MEANING OF THE EIKONAL EQ.

NOTICE THAT WE TOOK AN INITIAL CONDITION $u = \text{const}$ ON THE CURVE $x^2 + y^2 = 1$.

THE LEVEL CURVES OF $u(x,y)$ ARE OTHER CIRCLES CENTERED AROUND THE ORIGIN



THEY CAN BE INTERPRETED AS WAVE FRONTS IN GEOMETRIC OPTICS.

THE EIKONAL EQUATION ARISES IN THE GEOMETRIC OPTICS APPROXIMATION TO THE THEORY OF LIGHT (OR OTHER WAVES).

IN THIS INTERPRETATION WE LOOK AT A MONOCHROMATIC WAVE OF FREQUENCY ω, MOVING THROUGH SPACE.

ASSUMING THE WAVE HAS FORM

$$A(x,y,t) = e^{i(\omega t - u(x,y))},$$

THE WAVE FRONTS ≡ POINTS WHERE THE WAVE HAS A CONSTANT PHASE, ARE THE LEVEL CURVES OF $u(x,y)$.

THE WAVE EQUATION $\left[\frac{\partial^2}{\partial t^2} - c^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right] A = 0$,

WHERE c = SPEED OF PROPAGATION,

GIVES:

$$\left[-\omega^2 + c^2\left(u_x^2 + u_y^2\right) + c^2 i\left(u_{xx} + u_{yy}\right)\right] = 0$$

THE EIKONAL APPROXIMATION IS $|u_{xx} + u_{yy}| \ll u_x^2 + u_y^2$

THEREFORE IN THIS APPROXIMATION WE NEGLECT THE LAST TERM AND WE GET THE

EIKONAL EQ:

$$\boxed{u_x^2 + u_y^2 = \frac{\omega^2}{c^2}}$$

THIS ALSO HOLDS IF WE ARE IN A MEDIUM WHERE c IS NOT A CONSTANT BUT DEPENDS ON THE POINT, $c \to c(x,y)$, IN THIS

CASE WE GET

$$u_x^2 + u_y^2 = \frac{\omega^2}{c^2(x,y)}$$

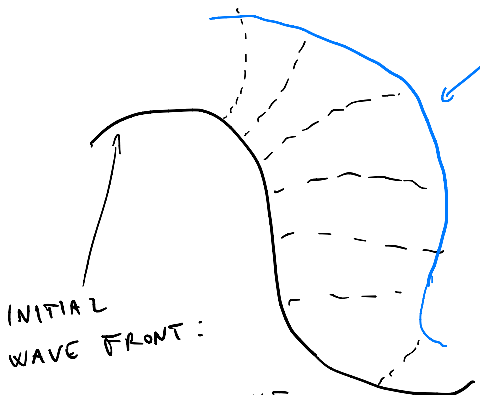
NOTE: ω CAN BE SCALED AWAY BY MULTIPLYING u FOR A CONSTANT.

GEOM. OPTICS INTERPRETATION:

WAVE FRONTS ARE FORMED BY (LIGHT)-RAYS.

THE RAYS ARE PERPENDICULAR TO ALL WAVE FRONTS, AND MOVE WITH VELOCITY $c(x,y)$ IN THE MEDIUM.

WAVE FRONTS ARE OBTAINED AS THE CURVES FORMED BY LIGHT RAYS AS THEY TRAVEL.



POINTS WHERE WE ASSUME THE WAVE HAS CONSTANT PHASE K.

WAVE FRONT AFTER TIME t:

$\equiv$ POINTS WHERE THE WAVE HAS THE SAME PHASE K AFTER TIME t.

$\equiv$ CURVE WHERE THE LIGHT RAYS ARRIVE AFTER TIME t.

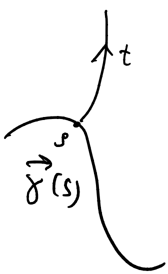
THE PHYSICAL INTUITION ABOVE IS ENOUGH TO DERIVE THE EIKONAL EQ.

CONSIDER AN INITIAL CURVE $\vec{\gamma}(s) = (x_0(s), y_0(s))$ WHERE THE WAVE HAS CONSTANT PHASE $= K$ AT TIME $t=0$.

$$A(x,y,t) = e^{i(\omega t - u(x,y))} \quad \rightarrow \quad u(\vec{\gamma}(s)) = K$$

AT TIME t, THE WAVE HAS THE SAME PHASE WHERE $u(x,y) = \omega t + K$

NOW WE IMPOSE THAT THIS LEVEL CURVE IS FORMED BY LIGHT-RAYS.



LET THE RAY EMITTED AT $\vec{\gamma}(s)$ HAVE TRAJECTORY $\vec{x}(s,t)$

WITH $\vec{x}(s, t=0) = \vec{\gamma}(s)$.

SINCE THE RAYS ARE ORTHOGONAL TO LEVEL CURVES OF u (i.e., WAVE FRONTS), THEY SATISFY:

- $$\frac{\partial}{\partial t}\vec{x}(s,t) = C(\vec{x}(s,t)) \cdot \frac{\nabla u(\vec{x}(s,t))}{\|\vec{\nabla} u(\vec{x}(s,t))\|}$$

(I.E., THE TANGENT VECTOR HAS LENGTH c, AND IS ORTHOGONAL TO THE LEVEL CURVES OF u

$\Updownarrow$

IT IS $\propto$ TO $\vec{\nabla} u$)

proportional

- AFTER TIME t, THEY FORM A NEW WAVE FRONT WITH:

$$u(\vec{x}(s,t)) = \omega t + K$$

DIFFERENTIATING THIS EQ. WE GET:

$$\frac{d}{dt} u(\vec{x}(s,t)) = \omega = (\vec{\nabla} u) \cdot \frac{\partial}{\partial t} \vec{x}$$

$$= c(\vec{x}(s,t)) \cdot \frac{(\vec{\nabla} u \cdot \vec{\nabla} u)}{\| \vec{\nabla} u \|}$$

$\hookrightarrow$ FROM THIS WE DEDUCE:

$$\vec{\nabla} u \cdot \vec{\nabla} u = \boxed{u_x^2 + u_y^2 = \frac{\omega^2}{c^2(x,y)}}$$

(WHICH IS THE EIKONAL EQUATION)

NOTICE:

- LIGHT-RAYS TRAVEL ALONG THE CHARACTERISTICS (ALTHOUGH THE "TIME" PARAMETER USED HERE IS NOT THE SAME AS "ℓ" USED ABOVE, BUT JUST RESCALED)

- THE FREEDOM TO CHOOSE AN INITIAL CONDITION GIVES US THE FREEDOM TO CHOOSE AN ARBITRARY CURVE $\vec{\gamma}(s)$ AS INITIAL WAVE FRONT

- WHEN $c=$ CONSTANTS, LIGHT RAY PATHS ARE STRAIGHT LINES, BUT IF $c(x,y)$ DEPENDS ON THE POINTS (x,y), (AS IN A MEDIUM WITH REFRACTIVE INDEX DEPENDING ON THE POINT), THEN THE PATHS ARE GENERALLY BENT.

2nd ORDER PDE's

WE WILL NOW STUDY SOME IMPORTANT (MOSTLY LINEAR) 2nd ORDER PDE's.

LINEAR CASE (1 D)

$$a(x,y)\,u_{xx} + 2b(x,y)\,u_{xy} + c(x,y)\,u_{yy} + d(x,y)\,u_x + e(x,y)\,u_y + f(x,y)\,u = g(x,y)$$

INHOMOGENEOUS TERM

(STRICTLY SPEAKING, IT IS LINEAR FOR $g=0$).

A SLIGHTLY MORE GENERAL CASE IS:

ALMOST LINEAR CASE:

$$\underbrace{a(x,y)\,u_{xx} + 2b(x,y)\,u_{xy} + c(x,y)\,u_{yy}}_{\text{PRINCIPAL PART (LINEAR)}} + \underbrace{F(x,y,u,u_x,u_y)}_{\text{CAN BE NONLINEAR}} = 0$$

EQUATIONS OF THIS KIND FALL INTO A CLASSIFICATION DEPENDING ON THEIR PRINCIPAL PART, WHICH DETERMINES TO A LARGE EXTENT THEIR BEHAVIOUR.

THE DISCRIMINANT IS DEFINED AS:

$$\boxed{\Delta(x,y) \equiv b^2(x,y) - a(x,y)\, c(x,y)}$$

IT CAN BE SHOWN THAT, FOR ANY NON-SINGULAR CHANGE OF VARIABLES

$$(x,y) \longmapsto (\xi,\eta) \qquad \text{WITH } J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

THE PDE TRANSFORMS INTO

$$A(\xi,\eta)\, u_{\xi\xi} + 2B(\xi,\eta)\, u_{\xi\eta} + C(\xi,\eta)\, u_{\eta\eta} + \Psi(\xi,\eta,u,u_\xi,u_\eta) = 0$$

WHERE THE DISCRIMINANT

$$\tilde{\Delta}(\xi,\eta) = B^2(\xi,\eta) - A(\xi,\eta)\, C(\xi,\eta)$$

HAS THE SAME SIGN AS $\Delta(x,y)$.

PRECISELY, WE FIND:

$$\left(B^2 - A\,C\right) = \left(b^2 - a\,c\right) \cdot \underbrace{\left(\xi_x \eta_y - \eta_x \xi_y\right)^2}_{= J^2 > 0}$$

THUS, THE SIGN OF THE DISCRIMINANT IS INVARIANT UNDER CHANGES OF COORDINATES

- NOTICE THAT A GENERIC EQUATION WITH COEFFICIENTS DEPENDING ON (x,y) MAY HAVE A DIFFERENT SIGN OF $\Delta(x,y)$ IN DIFFERENT REGIONS OF THE (x,y) PLANE

BASED ON THE SIGN OF $\Delta(x,y)$ (IN A CERTAIN REGION) WE CALL THE EQUATION:

- HYPERBOLIC WHEN $\Delta(x,y) > 0$
- PARABOLIC WHEN $\Delta(x,y) = 0$
- ELLIPTIC WHEN $\Delta(x,y) < 0$

THERE ARE VERY IMPORTANT REPRESENTATIVES OF THESE THREE CLASSES.

* $\boxed{\text{THE WAVE EQUATION}}$

$$u_{tt} - v^2 u_{xx} = 0$$ (HYPERBOLIC)

$a = 1$ $\qquad \Delta = b^2 - ac > 0$

$c = -v^2$

* **THE HEAT EQUATION** (PARABOLIC)

$$u_t - \alpha\, u_{xx} = 0 \quad , \quad \alpha > 0$$

$a = 0$

$b = 0$

$c = -\alpha$

$$\Delta = b^2 - ac = 0$$

* **THE LAPLACE EQUATION** (ELLIPTIC)

$$u_{xx} + u_{yy} = 0$$

$a = 1$

$c = 1$

$b = 0$

$$\Delta = b^2 - ac < 0$$

IN FACT, ALMOST-LINEAR EQUATIONS OF HYPERBOLIC, PARABOLIC OR ELLIPTIC TYPE CAN BE BROUGHT TO A FORM CLOSE TO THE WAVE, HEAT AND LAPLACE EQ'S, RESPECTIVELY.

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$*$ • IF $\Delta(x,y) > 0$ AROUND (x_0, y_0), THEN THERE IS A LOCAL CHANGE OF VARIABLES

$(x,y) \mapsto (\xi, \eta)$ AROUND (x_0, y_0),

SUCH THAT THE EQUATION BECOMES:

$$\left(\partial_\xi^2 - \partial_\eta^2\right) u = F(\xi, \eta, u, u_\xi, u_\eta)$$

(FOR SOME FUNCTION F).

• IF $\Delta(x,y) = 0$ AROUND (x_0, y_0), THERE IS A LOCAL C. OF COORDINATES BRINGING THE PDE TO THE FORM:

$$u_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta)$$

• IF $\Delta(x,y) < 0$, THERE IS A LOCAL C. OF COORDINATES TRANSFORMING THE

PDE INTO:

$$\left(\partial_\xi^2 + \partial_\eta^2\right) u = H\left(\xi, \eta, u, u_\xi, u_\eta\right)$$

THE PREVIOUS ARE CALLED "CANONICAL FORMS" FOR THE EQ.

- NOTICE THAT THE EQ. CAN BE BROUGHT TO ONLY ONE OF THE THREE FORMS, DEPENDING ON ITS DISCRIMINANT.

(in a given coordinate region where the sign is constant)

MOREOVER, IF THE PDE HAS CONSTANT COEFFICIENTS AND ONLY TERMS WITH TWO DERIVATIVES:

$$a\, u_{xx}(x,y) + 2b\, u_{xy}(x,y) + c\, u_{yy}(x,y) = 0$$

THEN THERE IS 1 LINEAR CHANGE OF COORDINATES:

$$\xi = \alpha x + \beta y$$

$$\eta = \gamma x + \delta y$$

WITH $\alpha\delta - \beta\gamma \neq 0$

WHICH TRANSFORMS IT INTO THE WAVE, HEAT OR LAPLACE EQUATION, RESPECTIVELY, DEPENDING ON WHETHER $\Delta = b^2 - ac$ IS > 0, $= 0$, OR < 0.

Thus, in this case it is really just the wave, hear or Laplace equation in disguise.

This (together with their physical applications) justifies why in the following we concentrate in detail on these 3 important cases.