1 Exercises on 1st order PDEs and Burgers equation

1.1 Burgers equation and generalisations

Ex. 1 . A slightly different Burgers. **Note:** The first 3 questions in this problem can be answered using the material in Lecture 10, while the last question 4) requires the material in Lecture 20.

Consider the generalised inviscid Burgers equation:

$$u_t + u^2 u_x = 0. \tag{1.1}$$

1) Write a solution for the equation with the method of characteristics.

2) Do the solutions to this equation develop gradient singularities with time evolution?

   Hint: Write the solution in implicit form, as was done in the lecture for the inviscid Burgers equation, and starting from that formula deduce an expression for u_x in terms of the initial data. Study if this expression blows up.

3) If the answer to the question above is yes, compute the time it takes for a singularity to form, given the initial profile $u(x, t = 0) = e^{-x^2}$. Compute the location where this first singularity occurs in the (x, t) plane.

4) Consider the equation above as a conservation law:

$$\partial_t(u) + \partial_x(\frac{u^3}{3}) = 0. \tag{1.2}$$

   Consider the discontinuous initial condition: $u(x, t = 0) = \Theta(-x)$, where $\Theta(x)$ is the step function. How does this initial condition evolve, if we want to preserve the integral form of the conservation law?

   Hint: Discontinuous solutions to conservation laws evolve according to the (appropriate version) of the Rankine-Hugoniot constraints discussed in the lectures. In the case of a step function, the solution preserves its shape but moves with a speed given by the Rankine-Hugoniot conditions. Derive the appropriate form of this condition to compute the speed of the shock front.

1.2 Initial value problems and general solutions for linear equations

Instructions: Solve the following initial value problems, or state if they do not admit a solution and why. Write also a fully general solution (NOTE: this can be done since these 1st order PDE's are linear - using the method in Lecture 9).

Hint: To check that the initial condition problem can have a solution, check that the Cauchy curve is not parallel to the characteristics.

Ex. 2 Solve:

$$u_x + yu_y = 2u, \tag{1.3}$$

with initial condition $u(1, s) = s$, $s \in \mathbb{R}$.

Find a formula for the general solution of this PDE and check the solution you found to the above IVP is correct.

Ex. 3 Solve:

$$u_x + u_y = u^2, \tag{1.4}$$

with initial condition $u(s, 0) = s^2$, $s \in \mathbb{R}$.

Find also a formula for the general solution.

Ex. 4 Solve:

$$xu_x + (y + x^2)u_y = u, \tag{1.5}$$

with initial condition $u(2, s) = s - 4$, $s \in \mathbb{R}$.

Find also a formula for the general solution.

Solution: For the special solution with this boundary condition you should find $u(x, y) = y - x^2$. For the general solution you should find $u(x, y) = xF\left(\frac{y-x^2}{x}\right)$, with F an arbitrary function.

Ex. 5 . In 3 variables:

- Solve (if possible) the initial value problems:

$$zu_x + yzu_z = 0 \tag{1.6}$$

  with initial condition $u(x, y, 1) = x^y$.

- Consider the different equation:

$$zu_x + yzu_y = 0 \tag{1.7}$$

  with the same initial condition $u(x, y, 1) = x^y$.

If the problem does not have a solution, explain why.

1.3 More complicated equations: quasi-linear and fully nonlinear

Hint: Use the method of characteristics, see Lectures 9-11 (if needed, use the version of the method for fully nonlinear equations). For all these problems, check that the initial condition allows for a solution, i.e., the Cauchy curve is not parallel to the characteristics.

Ex. 6 Solve:

$$u_x u_y = 2, \tag{1.8}$$

with initial condition $u(s, s) = 3s$, $s \in [0, 1]$.

Ex. 7 Solve:

$$u_x u_y = u, \tag{1.9}$$

with initial condition $u(s,1) = s,\ s \in [0,1]$.

Ex. 8 Solve (if possible) the following initial value problems for the PDE:

$$u u_x + u^2 u_y = u, \tag{1.10}$$

1) With initial condition: $u(s,1) = s,\ s \in [0,1]$.

2) With initial condition: $u(s,s) = 1,\ s \in [0,1]$.

Note: One of these problems does not admit a solution: which one and why?

Ex. 9 Solve:

$$u_t + u u_x = u, \tag{1.11}$$

with initial condition $u(x,t=0) = x,\ x \in [0,1]$.

Ex. 10 - Initial value problem, both specific and abstract Consider the PDE:

$$u_x + \frac{1}{2} u_y^2 = 1, \tag{1.12}$$

- Find the solution with the method of characteristics, corresponding to the initial condition $u(0,y) = y^2$. In this case, you should be able to write $u(x,y)$ explicitly.
- Write a solution with the method of characteristics, corresponding to the initial condition $u(0,y) = U_0(y)$. Does the solution ever become multi-valued? Specify if this depends on $U_0(y)$ and how.
- Show that u_x and u_y do *not* blow up for the solution, for any initial condition $U_0(y)$.

Hint: For the first point, the answer is $u(x,y) = x + \frac{y^2}{1+2x}$.

Observation: Notice that the solution can become multi-valued even without u_x or u_y blowing up. In the present case, what happens is that u_{yy} blows up, and after this u_y, rather than u, becomes discontinuous.

Ex. 11 - Eikonal equation with variable propagation speed Consider light propagating in a 2D material such that light travels with speed $c(x,y) = y$ proportional to the distance from the x-axis. In the geometric optics approximation, the phase of a wave with unit frequency travelling in the medium satisfies the eikonal equation:

$$u_x^2 + u_y^2 = \frac{1}{y^2}, \tag{1.13}$$

Consider the evolution of the initial wave front $u(x = 0, y) = 0,\ y > 0$ (i.e. the positive y-axis). What is the shape of the other wave fronts (i.e., level curves of u) in the (x, y) plane?

What is the path travelled by light rays in the medium?

Hint: As explained in the lecture, this is just the path traced by the characteristic curves for this equation.