

INHOMOGENEOUS BOUNDARY CONDITIONS (AND INHOMOGENEOUS WAVE EQ. ON THE INTERVAL)

WHEN WE HAVE INHOMOGENEOUS BOUNDARY CONDITIONS, THE GENERAL PRINCIPLE IS THE FOLLOWING: (CALLED "METHOD OF SHIFTING THE DATA")

- BREAK THE SOLUTION INTO

$$u(x, t) = \underline{u_b(x, t)} + u_0(x, t),$$

WHERE $u_b(x, t)$ IS A KNOWN FUNCTION SATISFYING THE INHOMOG. B. C.'S.

CASE A):

IF POSSIBLE, u_b SHOULD SATISFY THE WAVE EQ. ON ITS OWN

↳ THEN $u_0(x, t)$ SOLVES THE WAVE EQ.

WITH HOM. BOUNDARY CONDITIONS

AND CAN BE FOUND WITH THE METHODS DISCUSSED BEFORE.

CASE B): IF u_b DOES NOT SOLVE THE WAVE EQ., THEN u_0 SOLVES THE INHOMOGENEOUS W. EQ., WITH HOMOGENEOUS B. C.'S.

WE WILL SEE HOW TO TREAT THE
MORE COMPLICATED CASE B LATER.

THE METHOD FOR CASE A CAN BE
ILLUSTRATED WITH AN EXAMPLE.

EXAMPLE:

CONSIDER THE WAVE EQ. WITH THE FOLLOWING
BOUNDARY AND INITIAL CONDITIONS:

$$u_{tt} - c^2 u_{xx} = 0 \quad 0 \leq x \leq \pi, \quad t \geq 0.$$

$$\text{B.C.} \quad \begin{cases} u(0, t) = -1 \\ u_x(\pi, t) = 2 \end{cases} \quad \text{I.C.} \quad \begin{cases} u(x, 0) = \sin\left(\frac{x}{2}\right) + 2x - 1 \\ u_t(x, t)|_{t=0} = -2 \sin\left(\frac{3x}{2}\right) \end{cases}$$

↑ INHOMOGENEOUS BECAUSE $\neq 0$

IN THIS CASE WE CAN SUBTRACT A LINEAR
FUNCTION

$$u_b(x, t) = -1 + 2x$$

$$\text{CLEARLY: } (\partial_t^2 - c^2 \partial_x^2) u_b = 0 \quad (\text{SOLVES THE WAVE EQ.})$$

$$\text{AND} \quad \begin{cases} u_b(0, t) = -1 \\ \partial_x u_b(x, t)|_{x=\pi} = 2 \end{cases} \quad \text{SATISFIES B.C.'S}$$

THEN, DEFINING $u(x, t) \equiv u_b(x, t) + u_0(x, t)$,

WE HAVE THAT $u_0(x, t)$ SATISFIES:

$$(\partial_t^2 - c^2 \partial_x^2) u_0(x, t) \quad (\text{WAVE EQ.})$$

WITH HOMO G. B.C. $\begin{cases} u_0(0, t) = 0 \\ \partial_x u_0(x, t)|_{x=\pi} = 0 \end{cases}$
(DIRICHLET-NEUMANN TYPE)

AND REDEFINED INITIAL CONDITION:

$$\text{IC: } \begin{cases} u_0(x, t=0) = u(x, 0) - u_b(x, 0) = \sin\left(\frac{x}{2}\right) \\ \partial_t u_0(x, t)|_{t=0} = \partial_t u(x, t)|_{t=0} - \partial_t u_b(x, t)|_{t=0} = -2 \sin\left(\frac{3x}{2}\right) \end{cases}$$

THE SOLUTION USING FOURIER METHOD IS:

$$u_0(x, t) = \sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{t}{2} \cdot c\right) - \frac{4}{3c} \sin\left(\frac{3}{2}x\right) \sin\left(\frac{3}{2}t \cdot c\right)$$

[EXERCISE : MAKE SURE YOU AGREE WITH THE ABOVE SOLUTION! YOU SHOULD FIND IT WITH THE FOURIER METHOD FOUND IN THE PREVIOUS LECTURE. IN THIS CASE THE FOURIER COEFFICIENTS ARE ALREADY OBVIOUS, SO YOU DO NOT NEED TO COMPUTE THE INTEGRALS DEFINING THE FOURIER COEFFICIENTS, BECAUSE THE INITIAL DATA ARE ALREADY DECOMPOSED IN THE RELEVANT BASIS.]

THEREFORE, THE SOLUTION TO THE ORIGINAL PROBLEM IS:

$$u(x, t) = -1 + 2x + \sin\left(\frac{x}{2}\right) \cos\left(\frac{tc}{2}\right) - \frac{4}{3c} \sin\left(\frac{3x}{2}\right) \cdot \sin\left(\frac{3tc}{2}\right).$$

WHAT ABOUT THE CASE WHEN WE CANNOT FIND $u_b(x, t)$ SATISFYING THE WAVE EQ. ? (CASE B).

FOR EXAMPLE, THIS HAPPENS IN MOST CASES WHEN WE HAVE TIME-DEPENDENT BOUNDARY CONDITIONS

FOR INSTANCE, CONSIDER:

$$u_{tt} - c^2 u_{xx} = 0 \quad 0 \leq x \leq L, \quad t \geq 0.$$

$$\text{B.C.: } \begin{cases} u(0, t) = g(t) \\ u(L, t) = h(t) \end{cases} \quad \text{I.C.: } \begin{cases} u(x, 0) = \varphi(x) \\ \partial_t u(x, t)|_{t=0} = \psi(x) \end{cases}$$

↳ IN THIS CASE, WE STILL USE THE TRICK OF REDEFINING $u(x, t) = u_b(x, t) + u_0(x, t)$

WITH $u_b(x, t) = g(t) + \frac{x}{L} (h(t) - g(t))$

↑
CHOSEN IN SUCH A WAY THAT IT SATISFIES B.C.'S!

NOW WE HAVE THAT :

$$\begin{cases} u_0(0, t) = 0 \\ u_0(L, t) = 0 \end{cases} \quad \swarrow \searrow \quad \text{HOMOGENEOUS B.C. FOR } u_0$$

$$\text{I.C.: } \begin{cases} u_0(x, 0) = \varphi(x) - g(0) - \frac{x}{L} (h(0) - g(0)) \\ \left. \partial_t u_0(x, t) \right|_{t=0} = \psi(x) - g'(0) - \frac{x}{L} (h'(0) - g'(0)) \end{cases}$$

SHIFTED I.C. FOR u_0

HOWEVER, NOW u_0 SATISFIES THE
INHOMOGENEOUS WAVE EQ.

$$\partial_t^2 u_0(x, t) - c^2 \partial_x^2 u_0(x, t) = F(x, t)$$

$$\text{WITH } F(x, t) = -g''(t) - \frac{x}{L} (h''(t) - g''(t))$$

SO NOW THE PROBLEM IS REDUCED TO
SOLVING THE INHOMOGENEOUS W.EQ.

(WITH HOMOG. B.C.'S). THIS IS WHAT WE
TURN TO NEXT...

SOLVING THE INHOMOGENEOUS WAVE EQ. WITH HOMOG. B.C.'S

THIS CAN BE DONE WITH A SIMPLE GENERALISA-
TION OF THE FOURIER METHOD.

NOTE THAT, WHILE WE ILLUSTRATE IT FOR
THE WAVE EQ., THIS CAN BE USED ALSO
FOR ANY LINEAR PDE, IN PARTICULAR
FOR THE HEAT EQUATION WHICH WE STUDY
NEXT (SEE EXERCISES!)

WE EXPLAIN THE DIRICHLET-DIRICHLET CASE
(IT IS ANALOGOUS WITH OTHER B.C.'S)

$$u_{tt} - c^2 u_{xx} = F(x, t)$$

$$IC: \begin{cases} u(x, t=0) = \varphi(x) \\ \partial_t u(x, t) \big|_{t=0} = \psi(x) \end{cases}$$

$$BC: \begin{cases} u(0, t) = u(L, t) = 0 \end{cases}$$

WE BREAK THE PROBLEM INTO

$$u(x, t) = u_h(x, t) + v(x, t)$$

 HOMOGENEOUS

- u_h SOLVES THE HOMOGENEOUS PROBLEM

$$(\partial_t^2 - c^2 \partial_x^2) u_h = 0$$

WITH B.C. :
$$\begin{cases} u_h(0, t) = u_h(L, t) = 0 \end{cases}$$

$$\text{I.C. : } \begin{cases} u_h(x, t=0) = \varphi(x) \\ \partial_t u_h(x, t) \big|_{t=0} = \psi(x) \end{cases}$$

↳ WE FIND u_h WITH THE STANDARD FOURIER METHOD

- $v(x, t)$ SOLVES THE INHOMOGENEOUS PROBLEM WITH ZERO INITIAL DATA AND B.C.

$$(\partial_t^2 - c^2 \partial_x^2) v = F(x, t)$$

$$\text{I.C. : } \begin{cases} v(x, 0) = 0 \\ \partial_t v(x, t) \big|_{t=0} = 0 \end{cases}$$

$$\text{B.C. : } \begin{cases} v(0, t) = v(L, t) = 0 \end{cases}$$

TO FIND $V(x, t)$, WE ASSUME A DECOMPOSITION

OF THE FORM:

$$V(x, t) = \sum_{n=1}^{\infty} \sin(k_n x) \cdot V_n(t)$$

$$k_n = \frac{n\pi}{L}$$

THE x -PART IS CHOSEN
TO SATISFY THE B.C.'S:
IT IS THE SAME AS FOR
THE HOMOGENEOUS CASE,

SO, FOR E.G. THE NEUMANN-NEUMANN
PROBLEM IT WOULD BE A
COSINE-SERIES, ETC....

GENERIC TIME-
DEPENDENT
COEFFICIENTS:

NOW WE NEED
TO FIND THEM

DECOMPOSE THE SOURCE TERM $F(x, t)$

USING THE SAME BASIS:

$$F(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) \cdot f_n(t)$$

KNOWN FUNCTIONS

THEN THE PDE GIVES A SYSTEM OF
ODE'S FOR $V_n(t)$!

IN FACT $(\partial_t^2 - c^2 \partial_x^2) V = F(x, t)$

GIVES :

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) \cdot \left[\ddot{V}_n(t) - c^2 \left(\frac{n\pi}{L}\right)^2 V_n(t) \right] =$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) \cdot f_n(t)$$

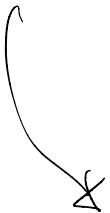
↳ IT MUST HOLD TERM BY TERM :

$$\ddot{V}_n(t) - c^2 \left(\frac{n\pi}{L}\right)^2 V_n(t) = f_n(t)$$

THESE ODE'S HAVE INITIAL CONDITION :

$$V_n(0) = \dot{V}_n(0) = 0$$

TO GUARANTEE $V(x, t=0) = \partial_t V(x, t) \Big|_{t=0} = 0$.



THIS ODE IS A SECOND ORDER ODE WITH
CONSTANT COEFFICIENTS AND A INHOMOGENEOUS TERM.

THE HOMOGENEOUS EQUATION (i.e., IF WE SET $f_n \rightarrow 0$)
HAS SOLUTIONS $\cos(\omega_n t)$ AND $\sin(\omega_n t)$,

$$\text{WITH } \omega_n = C \frac{n\pi}{L}.$$

AS WE SAW EARLIER IN THE COURSE, WE CAN
NOW BUILD A INHOMOGENEOUS SOLUTION USING
THE METHOD OF VARIATION OF CONSTANTS.

THE RESULT HAD AN EXPLICIT FORM IN TERMS
OF THE TWO HOMOGENEOUS SOLUTIONS.

USING THE FORMULA DEDUCED EARLIER,

THE GENERAL SOLUTION TO THE ODE FOR
 $v_n(t)$ IS:

$$v_n(t) = a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$$

$$- \int_0^t \frac{\begin{vmatrix} \cos \omega_n t & \sin \omega_n t \\ \cos \omega_n s & \sin \omega_n s \end{vmatrix}}{W(s)} f_n(s) ds,$$

WITH $W(s) = \begin{vmatrix} \cos \omega_m s & \sin \omega_m s \\ \omega_m \sin \omega_m s & -\omega_m \cos \omega_m s \end{vmatrix} = -\omega_m$

$$\hookrightarrow V_m(t) = a_m \cos \omega_m t + b_m \sin \omega_m t + \frac{1}{\omega_m} \int_0^t \sin \omega_m (s-t) f_m(s) ds$$

TO SATISFY THE INITIAL CONDITIONS

$$V_m(0) = \dot{V}_m(0) = 0 \quad \text{WE SET: } a_m = b_m = 0$$

$$\hookrightarrow \text{THUS } V_m(t) = \frac{1}{\omega_m} \int_0^t \sin \omega_m (t-s) f_m(s) ds$$

IS THE SOLUTION TO THE PROBLEM

LET US NOW ILLUSTRATE THE METHOD WITH SOME EXAMPLES AND DISCUSS THE PHENOMENON OF RESONANCE.

EXAMPLE

$$u_{tt} = u_{xx}$$

WITH INITIAL CONDITIONS:

$$IC: \begin{cases} u(x, 0) = 0 \\ u_t(x, t) \Big|_{t=0} = A \frac{\omega x}{L} \end{cases}$$

AND BOUNDARY CONDS:

$$BC: \begin{cases} u(L, t) = A \sin \omega t \\ u(0, t) = 0 \end{cases}$$

• I TAKE A FUNCTION SATISFYING THE B.C.

THE SIMPLEST WAY IS TAKING A FUNCTION LINEAR IN x :

$$w(x, t) \stackrel{\text{def.}}{=} A \frac{x}{L} \sin \omega t.$$

THEN I STUDY $v = u - w$

v SATISFIES A INHOMOGENEOUS PDE:

$$v_{tt} - v_{xx} = \cancel{u_{tt} - u_{xx}} - w_{tt} + w_{xx} = -\omega^2 A \frac{x}{L} \sin \omega t,$$

WITH BC: $V(0, t) = V(L, t) = 0$

AND IC:
$$\begin{cases} V(x, 0) = 0 \\ V_t(x, t)|_{t=0} = 0 \end{cases}$$

▷ DIRICHLET-
DIRICHLET

TO SOLVE THE PROBLEM I LOOK FOR:

$$V(x, t) = \sum_{n=1}^{\infty} V_n(t) \sin\left(\frac{n\pi}{L} x\right)$$

↑ TAKES CARE OF B.C.'S!

WITH INITIAL COND'S:

$$V_n(0) = \dot{V}_n(0) = 0$$

THE PDE GIVES:

$$\sum_{n=1}^{\infty} \left(-\left(\frac{n\pi}{L}\right)^2 V_n(t) + \ddot{V}_n(t) \right) \sin\left(\frac{n\pi}{L} x\right) = -\frac{\omega^2 A x}{L} \sin \omega t$$



DECOMPOSING :

$$x = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) d_n,$$

$$x \in [0, L]$$

WHERE

$$d_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) x \, dx,$$

WE ARRIVE AT A SYSTEM OF ODE'S :

$$-\left(\frac{n\pi}{L}\right)^2 V_n(t) + \ddot{V}_n(t) = -\frac{\omega^2 A}{L} d_n \cdot \sin \omega t$$

THE SOLUTION WITH BOUNDARY CONDITIONS

$$V_n(0) = \dot{V}_n(0) = 0 \quad \text{is}$$

$$\begin{aligned} V_n(t) &= -\frac{\omega^2 A d_n}{n\pi} \int_0^t \sin\left(\frac{n\pi}{L}(s-t)\right) \sin \omega s \, ds \\ &= -\underbrace{A \omega^2 d_n}_{n\pi} \cdot \frac{L^2 \omega \sin\left(\frac{n\pi t}{L}\right) - L n\pi \sin(\omega t)}{(n^2 \pi^2 - L^2 \omega^2)} \end{aligned}$$

THE FULL SOLUTION TO THE PROBLEM IS:

$$u(x, t) = \underbrace{\frac{A x \sin \omega t}{L}}_{w(x) t} + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) v_n(t)$$

WITH $v_n(t)$ COMPUTED ABOVE.

(IN THIS PARTICULAR CASE, $v_n(t)$ CONTAINS

A COEFFICIENT

$$d_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) x \, dx = \frac{2L}{(n\pi)^2} (-1)^{n+1}.$$

IN THE EXAM YOU WILL BE REQUIRED ONLY TO WRITE THE INTEGRAL EXPRESSION OF FOURIER COEFFICIENTS, NOT TO COMPUTE THE INTEGRALS.)

APPLICATION - THE PHENOMENON OF RESONANCE

WHEN WE APPLY AN EXTERNAL FORCE (i.e., INHOMOGENEOUS TERM) TO THE WAVE EQUATION WITH FREQUENCY ω , WE CAN PUMP ENERGY INTO THE SYSTEM AND GENERATE FLUCTUATIONS WITH AMPLITUDE GROWING QUICKLY WITH TIME, WHEN THE FREQUENCY OF THE APPLIED FORCE IS CLOSE TO ONE OF THE FUNDAMENTAL FREQUENCIES OF THE SYSTEM. THIS IS CALLED RESONANCE.

TO SEE HOW THIS CAN HAPPEN CONSIDER:
A FORCE APPLIED TO A STRING OF THE FORM:

$$u_t - c^2 u_{xx} = A \sin \omega t \sin\left(\frac{n\pi x}{L}\right)$$

KEEPING BC: $\begin{cases} u(0, t) = u(L, t) = 0 \end{cases}$

LET US LOOK AT THE SOLUTION WITH

$$IC: \begin{cases} u(x, 0) = 0 \\ \left. \partial_t u(x, t) \right|_{t=0} = 0 \end{cases} .$$

USING EXACTLY THE SAME METHOD AS
ABOVE WE GET:

$$u(x, t) = \sin\left(\frac{n\pi x}{L}\right) \cdot C(t) ,$$

(IN THIS CASE THERE IS ONLY ONE WAVE NUMBER,
 n , IN THE GAME)

WITH :

$$C(t) = \frac{LA}{n\pi} \int_0^t \sin\left(\frac{n\pi}{L} \cdot (s - t)\right) \cdot \sin \omega s \, ds$$

$$= \frac{A}{\omega_n (\omega_n^2 - \omega^2)} \cdot \left[\omega_n \sin \omega t - \omega \sin \omega_n t \right]$$

$$\left(\omega_n = \frac{n\pi}{L} \right)$$

FOR $\omega \rightarrow \omega_n$ THE AMPLITUDE OF THE FLUCTUATIONS WE GET A GROWTH OF THE AMPLITUDE WITH TIME

THE ASYMPTOTIC EXPRESSION FOR $\omega \rightarrow \omega_n$ IS:

$$C(t) \sim -\frac{A}{2\omega_n} \left[\overset{\text{GROWING}}{t} \cos \omega_n t - \frac{\sin \omega_n t}{\omega_n} \right]$$

THIS IS THE PHENOMENON OF RESONANCE;

A RELATIVELY SMALL APPLIED OSCILLATING FORCE CAN HAVE ITS EFFECT AMPLIFIED STRONGLY WHEN ω IS CLOSE TO ONE OF THE CHARACTERISTIC FREQUENCIES.

- THIS PHENOMENON IS EXPLOITED BY MUSICIANS
e.g. VIOLIN PLAYERS.

- IT IS THIS MECHANISM THAT ALLOWS YOU TO MAKE A GLASS "SING" BY RUBBING A

FINGER ON ITS RIM IN A CIRCULAR MOTION (WITH THE RIGHT FREQUENCY!)

- RESONANCE CAN BE DANGEROUS FOR SUSPENDED BRIDGES, AND IN THE DESIGN OF BRIDGES ENGINEERS TAKE CARE OF AVOIDING THE RISK OF IT GOING INTO RESONANCE WITH THE WIND AND EVEN WITH PEOPLE'S STEPS.

(e.g. THE MILLENNIUM BRIDGE IN LONDON HAD TO BE "FIXED" AFTER ITS INITIAL OPENING, AS IT SHOWED VERY LARGE LATERAL OSCILLATIONS WHEN CROSSED BY MANY PEOPLE AT THE SAME TIME, IN AN EFFECT SIMILAR TO THE RESONANCE WE HAVE DESCRIBED.)

PROBLEMS WITH INHOMOGENEOUS BC'S AND INHOMOGENEOUS TERMS IN THE EQUATION

CONSIDER A GENERAL PROBLEM:

$$u_{tt} - c^2 u_{xx} = F(x, t)$$

$$\begin{cases} u(0, t) = a(t) \\ u(L, t) = b(t) \end{cases} \quad (BC)$$
$$IC: \begin{cases} u(x, 0) = \phi(x) \\ \partial_t u(x, t) \big|_{t=0} = \psi(x) \end{cases}$$

TO SOLVE IT WE BREAK IT INTO PIECES
AS FOLLOWS:

$$u = w + v$$

$$\hookrightarrow w = a(t) + b(t) \frac{x}{L}$$

→ THEN : V SOLVES :

$$V_{tt} - c^2 V_{xx} = \underbrace{f(x,t) - a''(t) + c^2 b''(t) \frac{x}{L}}_{\equiv g(x,t)}$$

WITH BC: $V(0, t) = V(L, t) = 0$

AND IC:
$$\begin{cases} V(x, 0) = \varphi(x) - a(0) - \frac{b(0)}{L} x \\ \left. \partial_t V(x, t) \right|_{t=0} = \psi(x) - a'(0) - \frac{b'(0)}{L} x \end{cases}$$

NOW BREAK IT ONCE MORE INTO

$$V = V_1 + V_2$$

WITH V_1 SOLVING THE HOMOGEN. EQ.
WITH THE FULL I.C.,

$$(\partial_{tt} - c^2 \partial_{xx}) v_1 = 0$$

$$\begin{cases} v_1(x, 0) = \varphi(x) - a(0) - \frac{b(0)}{L} x \\ \partial_t v_1 \big|_{t=0} = \psi(x) - a'(0) - \frac{b'(0)}{L} x \end{cases}$$

$$\begin{aligned} v_1(0, t) &= v_1(L, t) \\ &= 0 \end{aligned}$$

AND v_2 SOLVING THE INHOMOGENEOUS PROBLEM:

$$(\partial_{tt} - c^2 \partial_{xx}) v_2 = f(x, t)$$

WITH $v_2(x, 0) = 0 = \partial_t v_2 \big|_{t=0} = 0$ (IC)

$$v_2(0, t) = v_2(L, t) = 0 \quad (BC)$$