LAPLACE / POISSON EQUATION

AS MAIN EXAMPLE OF "ELLIPTIC" PDE WE STUDY LAPLACE EQUATION. (AND ITS INHOMOGENEOUS VERSION, POISSON EQ.).

LAPLACE $$\Delta^{(D)} u(\vec{x}) = 0 \quad , \quad \vec{x} \in \mathbb{R}^D$$

$$\Delta^{(D)} = \sum_{i=1}^{D} \frac{\partial^2}{\partial x_i^2}$$

POISSON $$\Delta^{(D)} u(\vec{x}) = -\rho(\vec{x})$$

NOMENCLATURE: SOLUTIONS TO LAPLACE EQ. ARE CALLED "HARMONIC FUNCTIONS".

APPLICATIONS:

* STATIONARY (i.e. TIME-INDEPENDENT) SOLUTIONS OF THE WAVE OR HEAT EQUATIONS ARE DESCRIBED BY LAPLACE EQ.

* IF WE HAVE A CONSERVATIVE FORCE OBEYING GAUSS LAW, THE ASSOCIATED POTENTIAL SATISFIES LAPLACE OR POISSON EQ.

IN FACT:

A CONSERVATIVE FORCE HAS THE FORM

$$\vec{F} = -\vec{\nabla} u \qquad u: \text{ POTENTIAL}$$

GAUSS LAW IS THE STATEMENT THAT THERE IS A CHARGE ASSOCIATED TO THE FORCE, SUCH THAT FOR EVERY CLOSED VOLUME Ω WE HAVE

FLUX OF $\vec{F}$ ACROSS THE BOUNDARY OF Ω = CHARGE Q INSIDE Ω

$$\int_{\partial\Omega} dS\, \vec{F}\cdot\vec{n} = Q \text{ INSIDE } \Omega$$

$$\|$$

(WHERE $\vec{n}$ = UNIT NORMAL VECTOR TO $\partial\Omega$)

$$\|$$

$$\int_{\partial\Omega} dS\, (-\vec{\nabla} u)\cdot\vec{n} = -\int_{\Omega} \vec{\nabla}\cdot(\vec{\nabla} u)\, dV$$

DIVERGENCE THEOREM

SO FOR EVERY VOLUME Ω:

$$-\int_{\Omega} \Delta^{(D)} u \; dV = Q \text{ INSIDE } \Omega$$

IF THE CHARGE HAS A DENSITY $\rho(\vec{x})$

$$\Rightarrow \quad -\int_{\Omega} \Delta^{(D)}_{\vec{x}} u(\vec{x})\, dV_{\vec{x}} = \int_{\Omega} \rho(\vec{x})\, dV_{\vec{x}}$$

$$\Rightarrow \quad \boxed{\Delta^{(D)}_{\vec{x}}\, u(\vec{x}) = -\rho(\vec{x})}$$

(IN THE LAST STEP, WE USED THE FACT THAT THE IDENTITY ABOVE WAS VALID FOR EVERY Ω: SO IT MUST HOLD AT THE LEVEL OF INTEGRANDS).

EXAMPLES OF CONS. FORCES WITH GAUSS LAW ARE THE GRAVITATIONAL AND ELECTRIC FORCE: SO THE LAPLACE / POISSON EQUATION APPEARS TO DESCRIBE THE STATIONARY SOLUTIONS FOR THE GRAVITATIONAL OR ELECTRIC POTENTIAL.

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* FINALLY ANOTHER APPLICATION IS THE DESCRIPTION OF <u>INCOMPRESSIBLE</u> <u>**AND** IRROTATIONAL FLUID FLOW</u>.

ONE OF EULER EQUATIONS IS:

$$\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \qquad \text{(CONTINUITY EQ.)}$$

WHERE ρ = DENSITY OF FLUID

$\vec{v}$: VELOCITY OF FLUID

FOR INCOMPRESSIBLE FLUIDS: $\rho = \rho_0 = \text{CONST.}$

IRROTATIONAL MEANS $\vec{\nabla} \wedge \vec{V} = 0$

WHICH IMPLIES THE EXISTENCE OF A POTENTIAL u SUCH THAT $\vec{V} = \vec{\nabla} u$.

USING $\rho = \text{CONST.}$, EULER EQUATION BECOMES $\vec{\nabla} \cdot \vec{V} = 0$

SINCE $\vec{V} = \vec{\nabla} u \implies \vec{\nabla} \cdot \vec{\nabla} u = \Delta u = 0$

LAPLACE EQ. IN D=2 AND RELATION WITH COMPLEX ANALYSIS

RECALL THAT A COMPLEX ANALYTIC FUNCTION

$f(z)$, $f: \mathbb{C} \to \mathbb{C}$ SATISFIES THE

CAUCHY-RIEMANN CONSTRAINTS ON ITS REAL AND IMAGINARY PARTS.

IF $\mathrm{Re}\, f \equiv u(z)$

$\mathrm{Im}\, f \equiv v(z)$

$$z \equiv x + iy \quad , \quad x, y \in \mathbb{R}$$

THEN AT EVERY POINT WHERE f IS ANALYTIC:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases} \qquad \text{(CAUCHY-RIEMANN EQS)}$$

FROM THOSE TWO EQUATIONS,

WE OBTAIN
$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} v\right)$$

THEREFORE
$$\Delta u = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0 \, !$$

SIMILARLY:
$$\Delta v = 0 \, !$$

↳ SO THE REAL AND IMAGINARY PARTS OF A COMPLEX ANALYTIC FUNCTION BOTH SATISFY LAPLACE EQUATION.

IT IS NOT TOO DIFFICULT TO PROVE THAT THE CONVERSE IS ALSO TRUE: GIVEN <u>ONE</u> REAL SOLUTION TO LAPLACE EQ., u SATISFYING $\Delta u = 0$, ONE CAN SOLVE CAUCHY-RIEMANN EQUATIONS FOR

V, AND IN THIS WAY CONSTRUCT A COMPLEX FUNCTION $f = u + i v$ WHICH IS ANALYTIC.

THIS LINK WITH COMPLEX ANALYSIS HAS AN IMMEDIATE SURPRISING CONSEQUENCE:

SINCE COMPLEX ANALYTIC FUNCTIONS ARE C^∞, WE LEARN THAT SOLUTIONS TO LAPLACE EQ. ARE INFINITELY DIFFERENTIABLE!

IN FACT, THIS IS TRUE IN ANY DIMENSION, NOT JUST IN $D=2$.

PARENTHESIS:

IN THIS PROPERTY, LAPLACE EQ. IS LIKE THE HEAT EQUATION. INSTEAD THE WAVE EQ. IS VERY DIFFERENT SINCE IT CAN ADMIT SOLUTIONS WHICH HAVE NON DIFFERENTIABLE POINTS, WHICH JUST TRAVEL WITH SPEED C.

THE NATURAL WELL-POSED PROBLEMS FOR THE LAPLACE EQUATION ARE BOUNDARY-VALUE PROBLEMS:

GIVEN A REGION $\Omega \subset \mathbb{R}^D$ A SUBREGION WITH BOUNDARY $\partial\Omega$

FIND u SATISFYING $\Delta_{\vec{x}} u(\vec{x}) = 0$ FOR $\vec{x} \in \Omega$

SUCH THAT $u(\vec{x})\Big|_{\vec{x} \in \partial\Omega} = \varphi(\vec{x})$

(DIRICHLET BOUNDARY CONDITION)

IN SOME PHYSICAL SITUATIONS, IT IS ALSO RELEVANT TO CONSIDER THE NEUMANN PROBLEM WITH BOUNDARY CONDITIONS ON $\partial_\perp u$.

IT CAN BE PROVED RIGOROUSLY THAT THE DIRICHLET BOUNDARY VALUE PROBLEM FOR LAPLACE EQ. IS WELL-POSED.

IN CONTRAST, IT CAN BE SHOWN THAT INITIAL VALUE PROBLEMS (i.e., IF WE TRY TO TREAT A COORDINATE LIKE "TIME") ARE IN GENERAL ILL-POSED. WE DISCUSSED AN EXAMPLE ON THIS IN CLASS (SEE THE RECORDING - HOWEVER THIS WILL NOT BE ASKED AT THE EXAM)

TAKEN FROM EXAMPLE 6.1 IN STAVROULAKIS & TERSIAN.

SOME MATHEMATICAL PROPERTIES OF SOLUTIONS TO LAPLACE EQ. (i.e., HARMONIC FUNCTIONS)

(WITHOUT PROOF)

THESE ARE VALID IN ANY DIMENSION

* HARMONIC FUNCTIONS ARE C^∞ ON OPEN SETS

* THE AVERAGE OF A HARMONIC FUNCTION OVER ANY SPHERE EQUALS ITS VALUE AT THE CENTER OF THE SPHERE

THIS IS THE "MEAN VALUE PROPERTY".

NOTE: IN THIS THEOREM, BY SPHERE WE MEAN THE HYPERSURFACE DEFINED BY $|\vec{x}| = R$,

SO IN 2D IT IS A CIRCLE

IN 3D A STANDARD 3D SPHERE,

etc...

EXPLICITLY, THE MEAN VALUE PROPERTY READS:

IN 2D:

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi \; u\left(x_0 + r\cos\varphi,\, y_0 + r\sin\varphi\right) = u(x_0, y_0)$$

$$\forall \; x_0, y_0, \; r$$

$\rightarrow \frac{1}{2\pi}$ COMES FROM DIVIDING BY LENGTH OF CIRCLE. $2\pi r$.

IN 3D:

$$\frac{1}{4\pi}\int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \; u\left(\vec{x}_0 + \vec{x}(r,\theta,\varphi)\right) = u(\vec{x}_0)$$

$$\forall \; \vec{x}_0, \; r.$$

WITH

$$\vec{x}(r,\theta,\varphi) = \left(r\sin\theta\cos\phi,\; r\sin\theta\sin\phi,\; r\cos\theta\right).$$

$\hookrightarrow \frac{1}{4\pi}$ COMES FROM DIVIDING BY SURFACE OF SPHERE $4\pi r^2$.

* CONSEQUENCE !

A SURPRISING CONSEQUENCE OF THE MEAN VALUE PROPERTY IS THAT A HARMONIC FUNCTION CANNOT HAVE ANY LOCAL MAXIMUM OR MINIMUM : OTHERWISE, CENTERING A SMALL SPHERE AROUND THE MAX/MIN, WE WOULD FIND A CONTRADICTION WITH THE MEAN VALUE PROPERTY.

THIS LEADS TO THE FOLLOWING IMPORTANT PROPERTY:

* MAX/MIN PRINCIPLE FOR LAPLACE EQUATION.

IF u IS A HARMONIC FUNCTION IN $\underbrace{\Omega}_{\text{A REGION}} \subset \mathbb{R}^D$,

THEN

$$\max_{\vec{x} \in \Omega} u(\vec{x}) = \max_{\vec{x} \in \partial\Omega} u$$

AND:

$$\min_{\vec{x}\in\Omega} u(\vec{x}) = \min_{\vec{x}\in\partial\Omega} u(\vec{x})$$

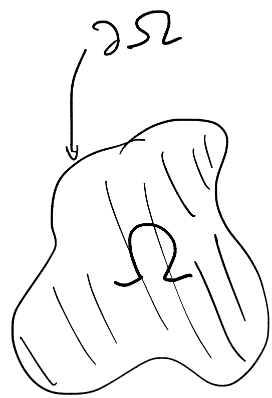
i.e. MAX AND MIN CAN ONLY BE REACHED AT THE BOUNDARY!

JUST LIKE FOR THE HEAT EQ.,
WE CAN USE THIS PROPERTY TO ESTABLISH NICE PROPERTIES OF CERTAIN PROBLEMS.

IN PARTICULAR WE SEE THAT:

THE DIRICHLET PROBLEM

$$\begin{cases} \Delta u = 0 & \text{IN } \Omega \subset \mathbb{R}^D \\ u\big|_{\partial\Omega} = \varphi \end{cases}$$



HAS A UNIQUE SOLUTION.

IN FACT, IF WE HAD TWO SOLUTIONS u_1, u_2, WE WOULD HAVE THAT

$$\max_{\Omega} (u_1 - u_2) = \max_{\partial\Omega} (u_1 - u_2) = 0$$

AND $$\min_{\Omega} (u_1 - u_2) = \min_{\partial\Omega} (u_1 - u_2) = 0$$

$\hookrightarrow$ SO $u_1 = u_2$.

MORE OVER THE DEPENDENCE ON THE BOUNDARY DATA $\varphi(x)$ IS CONTINUOUS.

WE ADAPT AGAIN THE SAME ARGUMENT SEEN FOR THE HEAT EQ.:

IF u_1 AND u_2 ARE THE TWO SOLUTIONS ASSOCIATED TO BOUND. VALUES φ_1, φ_2, THEN THEY SATISFY:

$$\max_{\Omega} |u_1 - u_2| \leq \max_{\partial\Omega} |\varphi_1 - \varphi_2|$$

SO THE VARIATION IN THE SOLUTIONS IS BOUNDED BY THE VARIATION IN THE BOUNDARY DATA.

THIS IMPLIES CONTINUITY.

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