Useful equations

These equations, if relevant for the test, will be distributed during the written exam.

Orthogonality properties for trigonometric functions: for $m, n \in \mathbb{N}$:

$$\frac{2}{L}\int_0^L \sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})dx = \delta_{m.n} \tag{0.1}$$

$$\frac{2}{L}\int_0^L \cos(\frac{n\pi x}{L})\cos(\frac{m\pi x}{L})dx = \delta_{m.n}\, 2^{\delta_{n,0}}, \tag{0.2}$$

$$\frac{2}{L}\int_0^L \cos(\frac{2n\pi x}{L})\sin(\frac{2m\pi x}{L})dx = \frac{2}{L}\int_0^L \cos(\frac{(2n+1)\pi x}{L})\sin(\frac{(2m+1)\pi x}{L})dx = 0. \tag{0.3}$$

Orthogonality for Bessel functions:

$$\int_0^1 dx\, x\, J_m(\mu_{m,i}x)\, J_m(\mu_{m,j}x) \propto \delta_{i,j}, \tag{0.4}$$

$$\int_0^1 dx\, x\, J_m(\nu_{m,i}x)\, J_m(\nu_{m,j}x) \propto \delta_{i,j}, \tag{0.5}$$

where $\mu_{m,i}$, $i = 1, \ldots, \infty$ are zeros of the Bessel function J_m: $J_m(\mu_{m,i}) = 0$, and $\nu_{m,i}$, $i = 1, \ldots, \infty$ are zeros of its first derivative: $J'_m(\nu_{m,i}) = 0$.

Bessel differential equation:

$$x^2 y''(x) + xy'(x) + (x^2 - \alpha^2)y(x) = 0. \tag{0.6}$$

For $\alpha \geq 0$, the solution with $y(x) \sim x^\alpha$ at $x \sim 0$ is $y(x) = J_\alpha(x)$ (Bessel function of the first kind).

Spherical Bessel differential equation:

$$x^2 y''(x) + 2xy'(x) + \left(x^2 - \ell(\ell+1)\right) y(x) = 0. \tag{0.7}$$

For $\ell \geq 0$, the solution with $y(x) \sim x^\ell$ at $x \sim 0$ is $y(x) = j_\ell(x) \equiv \sqrt{\frac{\pi}{2x}}\, J_{\ell+\frac{1}{2}}(x)$.

Orthogonality for spherical harmonics, and their definition:

$$\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, Y_\ell^m(\theta,\phi)\left(Y_{\ell'}^{m'}(\theta,\phi)\right)^* \propto \delta_{m,m'}\,\delta_{\ell,\ell'}. \tag{0.8}$$

Their definition is: $Y_\ell^m(\theta,\phi) = e^{im\phi} P_\ell^{|m|}(\cos\theta)$ ($|m| \leq \ell$, $m \in \mathbb{Z}, \ell \in \mathbb{N}$), with

$$P_\ell^{|m|}(x) \equiv \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{\frac{|m|}{2}}\,\frac{d^{\ell+|m|}}{dx^{l+|m|}}\left[(x^2-1)^\ell\right].$$

2D Laplace operator in polar coordinates

$$\Delta^{(2D)} = \partial_r^2 + \frac{\partial_r}{r} + \frac{\partial_\theta^2}{r^2}. \tag{0.9}$$

3D Laplace operator in polar coordinates

$$\Delta^{(3D)} = \partial_r^2 + 2\frac{\partial_r}{r} + \frac{\Delta_S}{r^2}, \tag{0.10}$$

where $\vec{x} = (x, y, z) = (r\cos\phi\sin\theta, r\sin\phi\sin\theta, r\cos\theta)$, and the angular part is:

$$\Delta_S = \partial_\theta^2 + \cot(\theta)\partial_\theta + \frac{\partial_\phi^2}{\sin^2\theta}. \tag{0.11}$$

The action of the angular part on spherical harmonics is: $\Delta_S \circ Y_\ell^m(\theta,\phi) = -\ell(\ell+1)Y_\ell^m(\theta,\phi)$.

Canonical form of P-symbol The "canonical form" of the Papperitz Riemann equation is

$$y = P\left\{ \begin{array}{cccc} & 0 & 1 & \infty \\ x & 0 & 0 & a \\ & 1-c & c-a-b & b \end{array} \right\} \tag{0.12}$$

and one of the solutions of this canonical ODE is:

$$y(x) =_2 F_1(a, b; c; x) \tag{0.13}$$

All other solutions can be found via transformations.

Wronskian formula for inhomogeneous solution of 2nd order linear ODE If y_1, y_2 are two independent solutions of the linear homogeneous ODE:

$$y'' + P(x)y' + Q(x)y = 0, \tag{0.14}$$

then

$$y_{\rm inh}(x) \equiv \int^x H(x,t)r(t)dt, \tag{0.15}$$

with

$$H(x,t) \equiv \frac{\left| \begin{array}{cc} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{array} \right|}{W(t)}, \quad W(t) \equiv \left| \begin{array}{cc} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{array} \right|, \tag{0.16}$$

satisfies

$$y_{\rm inh}'' + P(x)y_{\rm inh}' + Q(x)y_{\rm inh} = r(x). \tag{0.17}$$