Exercises on ODEs first part

0.1 Direct methods

Write the solutions to the following ODE problems for the function $y(x)$ using the methods discussed in the lectures. The methods are: separation of variables, the potential method, and methods for homogeneous and inhomogeneous linear equations, in particular variation of constants.

NOTE: one typo was corrected in Ex. 4.

Problems

1) Find the general solution to $xy'(x) = 1 + y(x)$;
2) Find the solution to $xy' - xy = y$ such that $y = 1$ for $x = 1$;
3) Write the general solution to $y'\left(y + x^2\right) + 2xy + \sin(x) = 0$.
4) Consider the ODE $y'' + 2y' + y = e^{-x}$. A particular solution is given by $y = \frac{x^2 e^{-x}}{2}$. Write the general solution.
5) Find the general solution to $y' - xy = 1$.
6) Solve $(1 + y^2) + xyy' = 0$ with $y = 0$ when $x = 5$.
7) Write in implicit form the general solution to $y'(xe^y + 1) + x^2 + e^y = 0$.
8) Find the solution to $y' + x^2 y = x^2$ such that $y = 3$ for $x = 0$.
9) Consider the ODE: $y'' + 2xy' - 2y = 0$. A solution is given by $y(x) = x$. Write the general solution.
10) Solve $(\cos(x) + 1)y' - (y + 1)\sin(x) - 2x = 0$, with $y(0) = 0$.

Solutions and hints:

1) Hint: separable equation. Solution: $y(x) = Ax - 1$.
2) Hint: separable. Solution: $y(x) = e^{-1+x}x$.
3) Hint: use the potential method. Solution: $y(x) = \pm\sqrt{A + x^4 + 2\cos(x) - 1} - x^2$.
4) Hint: we need to construct two solutions of the homogeneous equation, using the exponential ansatz. Solution: The two independent solutions are e^{-x} and xe^{-x} (Notice that the indices of the characteristic equation coincide). The general solution is $y(x) = \frac{x^2 e^{-x}}{2} + (A_1 + A_2 x)e^{-x}$.
5) Hint: use the variation of constants method. Solution: $y(x) = e^{\frac{x^2}{2}} + A\int_0^x e^{\frac{x^2 - s^2}{2}} ds$.

6) Hint: separable. Solution: $y(x) = \pm\frac{\sqrt{25-x^2}}{x^2}$.

7) Hint: use the potential method. Solution: the potential is (apart for additive constant) $U(x,y) = \frac{x^3}{3} + xe^y + y - 1$ and solutions are given in implicit form by $U(x,y) = A$.

8) Hint: (for instance) you can use the variation of constants method to find the particular solution to the inhomogeneous equation (but you can also probably guess it). Solution: $y(x) = 1 + 2e^{-\frac{x^3}{3}}$.

9) Hint: Construct a second independent solution by variation of constants. Solution: the second solution is obtained after solving the ODE with the ansatz $y_2(x) = a_2(x)x$. We find $y_2(x) = x\int^x e^{-s^2}\frac{ds}{s^2}$. The general solution is $y(x) = A_1x + A_2x\int^x e^{-s^2}\frac{ds}{s^2}$.

10) Hint: Potential method. Solution: $y(x) = \frac{x^2-\cos(x)+1}{\cos(x)+1}$.