1 Exercises on 2nd order PDEs

Note: In all these exercises, Fourier coefficients can be expressed as integrals. It is not required to compute the integrals explicitly.

NOTE: typos corrected in (1.29), (1.38).

1.1 Heat equation

Hint: When the problem has non-homogeneous boundary conditions, it is convenient to subtract an appropriate function in order to get a problem with homogeneous boundary conditions.

Ex. 1 (a)

Consider a bar of length L with heat diffusivity coefficient α. The bar's ends are kept at constant temperatures $u(0,t) = T_1$, $u(L,t) = T_2$. The initial temperature is $u(x, t = 0) = 4(L - x)^2 x^2$.

- What is the temperature distribution at $t = \infty$? (This can be answered without explicit calculations).
- Write the solution at generic times.

Solution To find the solution we split

$$u(x,t) = T_1 + x\frac{T_2 - T_1}{L} + v(x,t). \tag{1.1}$$

Now $v(x,t)$ has boundary conditions $v(0,t) = v(L,t) = 0$.

- Because of the boundary conditions, $v(x,t)$ will be expanded in a sine-series, which implies (for the heat equation) that its limit at large times is zero (we will see it explicitly below). Therefore we conclude that

$$\lim_{t\to\infty} u(x,t) = T_1 + x\frac{T_2 - T_1}{L}. \tag{1.2}$$

- The solution has the form

$$u(x,t) = T_1 + x\frac{T_2 - T_1}{L} + \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{L}x) e^{-\alpha(\frac{n\pi}{L})^2 t}, \tag{1.3}$$

where, matching the initial condition, we find

$$c_n = \frac{2}{L}\int_0^L dx \sin(\frac{n\pi}{L}x)\, v(x,0), \tag{1.4}$$

with $v(x,0) = 4(L-x)^2x^2 - T_1 - x\frac{T_2-T_1}{L}$.

Ex. 1 (b)

Consider the same setup as above, with initial condition a constant $u(x, t = 0) = T_0$, and the two ends both kept at constant temperature $T = 0$, i.e. $u(0, t) = u(L, t) = 0$. Write the solution at generic times as a Fourier series.

Solution In this case

$$u(x,t) = \sum_{n=1}^{\infty} c_n \ \sin(\frac{n\pi}{L}x)e^{-\alpha(\frac{n\pi}{L})^2 t}, \tag{1.5}$$

with

$$c_n = \frac{2}{L}\int_0^L dx \sin(\frac{n\pi}{L}x)\ T_0 = T_0\ \frac{2}{n\pi}\left(1-(-1)^n\right). \tag{1.6}$$

Ex. 2 - One-dimensional, different boundary conditions

A bar of length $L = \pi$ is initially at temperature $T = 0$ at time $t = 0$. At times $t > 0$, the left end of the bar is insulated, while the right end is kept at constant temperature $T = 50$. Find an expression for the time-dependent temperature distribution.

Hint: The fact that the left end is insulated means that there is no heat flow across this end. Remember that the heat flow is proportional to the gradient of the temperature.

Solution The boundary conditions are $u_x(0, t) = 0$, $u(\pi, t) = 50$.

Because the right BC is not homogeneous we subtract:

$$u(x,t) = v(x,t) + ax + b \tag{1.7}$$

such that $v_x(0, t) = 0$, $v(\pi, t) = 0$. We should choose $a = 0$, $b = 50$, so $u(x, t) = v(x, t) + 50$.

The initial condition $u(x, 0) = 0$ now becomes

$$v(x,0) = -50. \tag{1.8}$$

and $v_x(0, t) = v(\pi, t) = 0$. Expanding in the appropriate eigenfunctions we have

$$v(x,t) = \sum_{n=0}^{\infty} c_n \ \cos((n+\frac{1}{2})x)e^{-\alpha(n+\frac{1}{2})^2 t}, \tag{1.9}$$

where

$$c_n = -50\frac{2}{\pi}\int_0^{\pi} dx \ \cos((n+\frac{1}{2})x) = \frac{100(-1)^{n+1}}{\pi(n+\frac{1}{2})}. \tag{1.10}$$

So:

$$u(x,t) = 50 + \sum_{n=0}^{\infty} \frac{100(-1)^{n+1}}{\pi(n+\frac{1}{2})} \ \cos((n+\frac{1}{2})x)e^{-\alpha(n+\frac{1}{2})^2 t} \tag{1.11}$$

Ex. 3 - With time-dependent boundary conditions

At time $t = 0$, a bar of length L has uniform temperature $u(x, t = 0) = 0$, $0 \leq x \leq L$. For $t > 0$, the endpoints of the bar are heated such that

$$u(x = 0, t) = 3t, \quad u(x = L, t) = 4t, \tag{1.12}$$

for $0 \leq t \leq 1$. What is the temperature distribution at $t = 1$?

Hint: Do a subtraction in order to get a homogeneous boundary condition. Notice that in this case this will lead you to a non-homogeneous PDE to solve. You can solve it using a Fourier decomposition with time-dependent coefficients (the same trick that we used to solve the problem of the wave equation subject to an external force). Derive the ODE satisfied by the coefficients and solve it.

Solution Again we do a subtraction

$$u(x, t) = 3t + \frac{x}{L} t + v(x, t), \tag{1.13}$$

so that

$$v(0, t) = v(L, t) = 0. \tag{1.14}$$

The IC is in this case still

$$v(x, 0) = 0. \tag{1.15}$$

and now v satisfies the inhomogeneous PDE:

$$v_t - \alpha v_{xx} = -3 - \frac{x}{L}. \tag{1.16}$$

Now we try to solve with

$$v(x, t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L} x)\, c_n(t), \tag{1.17}$$

where the initial condition demands

$$c_n(0) = 0. \tag{1.18}$$

From the PDE (1.16), we deduce an ODE for the coefficients:

$$c_n'(t) + \alpha k_n^2 c_n(t) = d_n, \tag{1.19}$$

with

$$k_n \equiv \frac{n\pi}{L}, \tag{1.20}$$

where d_n is the coefficient of the decomposition $-3 - \frac{x}{L} = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L} x)\, d_n$ for $0 \leq x \leq L$, i.e.

$$d_n = -\frac{2}{L} \int_0^L (3 + \frac{x}{L}) \sin(\frac{n\pi}{L} x)\, dx. \tag{1.21}$$

To solve the ODE (1.19) we notice that the homogeneous equation has solution $e^{-\alpha t k_n^2}$. We can find a solution of the inhomogeneous equation by the variation of constants method. or we just notice that we can choose a constant solution, since the inhomogeneous term is independent on t. Indeed, a solution of the inhomogeneous equation is $c_{\text{inh}}(t) = \frac{d_n}{\alpha k_n^2}$, so $c_n(t) = A_n e^{-\alpha k_n^2 t} + d_n/\alpha k_n^2$, and matching the initial condition (1.18) we find

$$c_n(t) = \frac{d_n}{\alpha k_n^2} \left(1 - e^{-\alpha k_n^2 t}\right). \tag{1.22}$$

This completes our solution:

$$u(x,t) = 3t + \frac{x}{L}t + \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x)\, \frac{d_n}{\alpha k_n^2} \left(1 - e^{-\alpha k_n^2 t}\right). \tag{1.23}$$

with d_n defined in (1.21), and we can use this formula to evaluate the temperature at $t = 1$.

Ex. 4 - Sphere

Write the equations describing the cooling of a sphere of radius R, with initial uniform temperature $u(\vec{x}, t = 0) = T_0 > 0$, which is immersed in a space with uniform temperature $T_{\text{ext}} = 0$.

Hint: use spherical coordinates. See formulas at the end of the file for the relevant eigenfunctions (there are some simplifications given this initial distribution).

Solution The solution of this exercise is written in the notes' file in the chapter on the heat equation.

1.2 Wave equation

Ex. 5 - 1D

A string of length L, fixed at the points $x = 0$ and $x = L$ on the horizontal axis, has initially a displacement of the form $u(x, t = 0) = \sin^2(\frac{x\pi}{L})$, and its transversal velocity is $\frac{\partial}{\partial t} u(x,t)\big|_{t=0} = 0$. The endpoints of the strings are kept fixed. The propagation speed for waves on the string is v.

- Is the motion of the string for $t > 0$ periodic? If yes, what is its period (i.e., minimal time such that the motion repeats itself)?
- Derive an explicit expression for $u(x,t)$ for $t > 0$ (it can be written as an infinite Fourier series, which does not need to be summed. The coefficients should be defined explicitly, as integrals).

Hint: The first point can be answered both using the method of images and using the Fourier decomposition method. Try to deduce the answer using both methods.

Solution

- The motion of the wave equation in an interval domain is periodic. To compute the period, we notice that in the Fourier method the solution will be written in terms of functions with time periods $\frac{2\pi}{\omega_n}$, with $\omega_n = \frac{n\pi}{L}v$. These periods are all integer subdivisions of the period of the first harmonic, which is therefore the period of the full solution: the period is $2\pi/\omega_1 = \frac{2L}{v}$. Alternatively, we can think in terms of the method of images: the solution will be given by the evolution of an odd-extension of the solution, which is a function with space period $2L$. Since in the wave equation features of the solution propagate with speed v, to obtain the time period we can just divide the space period by v. We reobtain, indeed, $2L/v$.

- The form of the solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\frac{nx\pi}{L})\, \cos(\frac{n\pi}{L}vt), \tag{1.24}$$

  where we already used the condition that $u_t = 0$ at $t = 0$ to eliminate sines in the t variable.

  The coefficients (matching with initial condition) are

$$c_n = \frac{2}{L}\int_0^L dx \sin(\frac{nx\pi}{L})\, \sin^2(\frac{x\pi}{L}). \tag{1.25}$$

Ex. 6 - Vibrations of a drum

Consider a circular drum of radius R. The propagation speed for waves on the drum is v.

- What are the vibration frequencies of the drum?

- Suppose that at time $t = 0$ the displacement of the drum's membrane is $u(r, \theta, t = 0) = (r^2 - R^2)\cos\theta$, and its initial velocity is $\partial_t u(r, \theta, t = 0) = 0$. Write an explicit solution using the eigenfunctions method for the displacement of the membrane. The coefficients of the series can be defined as explicit integrals, but there is no need to compute the integrals or sum the series.

Hint: To recall the form of the eigenfunctions in this case, see notes at the end of the file.

Solution

- Recalling the form of the solution in polar coordinates (see below), the vibration frequencies can be $\omega_{m,i} = v\frac{\mu_{m,i}}{R}$, $m \in \mathbb{N}$, $i \in \mathbb{N}^+$, where $\mu_{m,i}$ are zeros of Bessel functions of the first kind with integer parameter: $J_m(\mu_{m,i}) = 0$ for $i = 1, 2, 3, \ldots$, $(m \in \mathbb{N})$.

- Since the initial condition contains the angular part $\cos\theta$, it is already decomposed in the angular part, with $m = 1$. This tells us that we should have a decomposition of the form

$$u(r,\theta,t) = \cos\theta \sum_{i=1}^{\infty} J_1(\mu_{1,i}\frac{r}{R})\ (c_i \cos(\omega_{1,i}t) + d_i \sin(\omega_{1,i}t))\,, \quad 0 \le r \le R, \tag{1.26}$$

with $\omega_{1,i}$ defined above.

From the initial conditions we read:

$$d_i = 0, i = 1, 2, 3, \ldots, \tag{1.27}$$

and

$$c_i = \frac{\int_0^R rdr J_1(\mu_{1,i}\frac{r}{R})\,(r^2 - R^2)}{\cdot\int_0^R rdr(J_1(\mu_{1,i}\frac{r}{R}))^2.}. \tag{1.28}$$

Ex. 7 - A different drum

Consider a second drum (with the same propagation speed v) which has the shape of a quarter of a disk, namely in radial coordinates, $0 \le r \le R$, $0 \le \theta \le \frac{\pi}{2}$.

- What are the vibration frequencies of the drum?
- Suppose that at time $t = 0$ the displacement of the drum's membrane is $u(r,\theta,t=0) = 0$, and its initial velocity is $\partial_t u(r,\theta,t=0) = \sin(2\theta)(r-R) + \sin(4\theta)(r-R)^2$. Write an explicit solution using the eigenfunctions method for the displacement of the membrane.

Solution

- In the new situation we have an additional boundary condition, namely $u(r,\theta,t)$ has to satisfy $u(r,0,t) = u(r,\frac{\pi}{2},t) = 0$. This means that for the angular part, the allowed eigenfunctions are no longer $\sin m\theta$, $\cos m\theta$, with $m \in \mathbb{N}$, but only $\sin(k\theta)$ with $k \in 2\mathbb{N}$ (which guarantees that at $\theta = \pi/2$ they vanish). Thus, the radial part will be given by Bessel function with indices k which can only be even integers. Thus, the frequencies of this drum are $v\frac{\mu_{k,i}}{R}$, with $k \in \mathbb{N}$, $i \in \mathbb{N}^+$ (which is a subset of the ones of the full circular drum).

  GENERALIZATION: Notice also that if we had a drum with the shape of a circle sector with angle α (rather than $\pi/2$), then the frequencies would be $v\frac{\mu_{k,i}}{R}$ with $k \in \mathbb{N}\frac{\pi}{\alpha}$, $i \in \mathbb{N}^+$.

- The solution with this initial conditions should have the form

$$u(r,\theta,t) = \sin(2\theta)\sum_{i=1}^{\infty} a_i J_2(\frac{\mu_{2,i}}{R}r)\ \sin(\omega_{2,i}t) + \sin(4\theta)\sum_{i=1}^{\infty} b_i J_4(\frac{\mu_{4,i}}{R}r)\ \sin(\omega_{4,i}t), \tag{1.29}$$

($\omega_{k,i} \equiv v\frac{\mu_{k,i}}{R}$), where we exploited that the initial condition was already decomposed in the angular variable, and also we used $u = 0$ at $t = 0$ to eliminate the cosines in the time dependence.

The coefficients, obtained matching with u_t at $t = 0$, are:

$$a_i = \frac{1}{\omega_{2,i}} \frac{\int_0^R rdr J_2(\mu_{2,i}\frac{r}{R})\,(r-R)}{\cdot\int_0^R rdr (J_2(\mu_{2,i}\frac{r}{R}))^2}, \tag{1.30}$$

$$b_i = \frac{1}{\omega_{4,i}} \frac{\int_0^R rdr J_4(\mu_{4,i}\frac{r}{R})\,(r-R)^2}{\cdot\int_0^R rdr (J_4(\mu_{4,i}\frac{r}{R}))^2}. \tag{1.31}$$

1.3 Laplace/Poisson equation

Ex. 8 - Dirichlet rectangle

A square metal plate of side π ($0 \le x,\ y \le \pi$), has its sides kept at the following temperatures:

$$T(x,0) = T(\pi,y) = 0, \quad T(0,y) = \sin y, \quad T(x,\pi) = 5\sin 2x - 7\sin 8x, \quad 0 \le x,\ y \le \pi. \tag{1.32}$$

What is the equilibrium temperature of the plate?

Solution The equilibrium temperature can be found by solving the Laplace equation (i.e., what remains of the heat equation after imposing time independence).

To solve the problem on this square domain, we need to break into two, $T(x,y) = u_1(x,y) + u_2(x,y)$, where u_1, u_2 solve Laplace equation with BC's:

$$u_1(0,y) = \sin(y),\ 0 \le y \le \pi,\ \text{and is zero on the other three sides}\ , \tag{1.33}$$

$$u_2(x,\pi) = 5\sin 2x - 7\sin 8x,\ 0 \le x \le \pi,\ \text{and is zero on the other three sides}\ . \tag{1.34}$$

Solutions are found with the eigenfunctions method as explained in the notes in the Laplace equation part.

Notice that here we do not have infinite sums because the data of the problem are already divided into a a finite number of eigenfunctions, so the other ones will never be involved. So, even though in general we would have $u_1(x,y) = \sum_n \sin ny\ (A_n \cosh nx + B_n \sinh nx)$, here we will have only a single term:

$$u_1(x,y) = \sin y\ (A\cosh x + B\sinh x) \tag{1.35}$$

and imposing that $u_1(\pi,y) = 0$ and $u_1(0,y) = \sin y$ we get:

$$u_1(x,y) = \sin(y)\ \frac{\sinh(\pi - x)}{\sinh(\pi)}. \tag{1.36}$$

Similarly we find:

$$u_2(x,y) = 5\sin(2x)\frac{\sinh(2y)}{\sinh(2\pi)} - 7\sin(8x)\frac{\sinh(8y)}{\sinh(8\pi)}, \tag{1.37}$$

and the full solution is $T_{\text{equilibrium}}(x,y) = u_1(x,y) + u_2(x,y)$.

Ex. 9 - Neumann rectangle

Consider the Laplace equation $u_{xx}+u_{yy}=0$ on a rectangular domain of sides L, M ($0 \le x, \le L$, $0 \le y \le M$), with Neumann boundary conditions

$$u_x(0,y) = u_x(L,y) = u_y(x,M) = 0, \quad u_y(x,0) = f(x). \tag{1.38}$$

Write a form for the solution using Fourier's method. Specify any consistency conditions that are required.

Solution The Neumann problem for Laplace's equation always has a consistency condition (see notes).

The consistency condition is (see notes)

$$\int_{\partial\Omega} \frac{\partial}{\partial n} u \, dS = 0, \tag{1.39}$$

where Ω is the domain, so in our case it becomes

$$\int_0^L dx f(x) = 0. \tag{1.40}$$

We will see why it is needed to solve the problem, also in the eigenfunctions method.

Since the solution has zero Neumann BC's on three sides, apart for the one at $y=0$, we use a decomposition of the form*

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} \cos(\frac{n\pi}{L}x) \left(A_n \cosh(\frac{n\pi}{L}y) + B_n \sinh(\frac{n\pi}{L}y)\right), \tag{1.41}$$

We still need to impose the BC's on the top and bottom sides. The top side gives

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} \cos(\frac{n\pi}{L}x) \; C_n \cosh(\frac{n\pi}{L}(y-M)), \tag{1.42}$$

and the condition on the bottom side gives

$$\sum_{n=1}^{\infty} \cos(\frac{n\pi}{L}x) \; C_n \; \frac{n\pi}{L} \sinh(\frac{n\pi}{L}(M)) = f(x). \tag{1.43}$$

The condition for $f(x)$ to be expandable in this cosine series *without the constant term* is precisely the condition (1.40). If and only if (1.40) is satisfied, then we have

$$C_n = \frac{2}{n\pi \, \sinh(\frac{n\pi}{L}(M))} \int_0^L dx f(x) \cos(\frac{n\pi}{L}x), \tag{1.44}$$

and this completes the solution.

*Notice that A_0 is an additive constant which we cannot fix (the Neumann problem has this ambiguity, i.e. the solution is defined up to an additive constant).

Ex. 10 - Disk

Consider Laplace equation $\Delta^{(2D)}u = 0$ in the interior of a disk of radius R, with boundary condition $u(R, \theta) = f(\theta)$ on the boundary of the disk.

- Using the Green's function method discussed in the lecture, write down explicitly the solution at an interior point $u(r, \phi)$, as a function of the boundary data.
- Derive an alternative form of the solution using the Fourier method.

Solution It is contained in the notes in the part on Laplace equation.

Ex. 11 - Annulus

Consider Laplace equation in the interior of an annulus region, namely the region defined by $R_1 < r < R_2$, $r = \sqrt{x^2 + y^2}$.

Consider the case $R_2 = 2$, $R_1 = 1$. Using a decomposition into eigenfunctions in radial coordinates, solve the following boundary value problem:†

$$\Delta^{(2D)}u = 0, \quad R_1 < r < R_2, \quad u(R_1, \theta) = \sin 2\theta, \quad u(R_2, \theta) = 0. \tag{1.45}$$

Solution In the second part of the previous question, we saw that in polar coordinates solutions to Laplace equation in the disk are a superposition of terms $\sum_{n=0}^{\infty} r^n \, (A_n \cos(n\theta) + B_n \sin(n\theta))$. This follows because r^n is the solution to the radial part of the Laplace equation, once the angular part is fixed. There is a second solution r^{-n} (or $\log r$ for $n = 0$), which is discarded because it is singular at $r = 0$.

However, now that we study an annular region there is no reason to exclude these solutions and indeed we need them in order to satisfy the extra boundary condition on the inner boundary.

So the ansatz now is:

$$u(r, \theta) = \sum_{n=1}^{\infty} r^n \, (A_n \cos(n\theta) + B_n \sin(n\theta)) + \sum_{n=1}^{\infty} r^{-n} \, (C_n \cos(n\theta) + D_n \sin(n\theta)) + A_0 + C_0 \log r, \tag{1.46}$$

for $R_1 < r < R_2$. The two boundary conditions give

$$\begin{aligned} \sin(2\theta) &= A_0 + C_0 \log R_1 + \sum_{n=1}^{\infty} \left[\left(A_n R_1^n + C_n R_1^{-n}\right) \cos(n\theta) + \left(B_n R_1^n + D_n R_1^{-n}\right) \sin(n\theta) \right], \\ 0 &= A_0 + C_0 \log R_2 + \sum_{n=1}^{\infty} \left[\left(A_n R_2^n + C_n R_2^{-n}\right) \cos(n\theta) + \left(B_n R_2^n + D_n R_2^{-n}\right) \sin(n\theta) \right]. \end{aligned}$$

† The boundary condition is simplified with respect to the original file, just to make the calculation shorter. One could solve the general case with exactly the same method, i.e. the inner and outer boundary conditions can be any functions.

We see that we have to set to zero all coefficients[‡] apart for the following ones:

$$B_2 = \frac{R_1^2}{R_1^4 - R_2^4}, \quad D_2 = -\frac{R_1^2 R_2^4}{R_1^4 - R_2^4}, \tag{1.47}$$

so the solution is

$$u(r,\theta) = \sin(2\theta)\left(\frac{R_1^2}{R_1^4 - R_2^4} r^2 - \frac{R_1^2 R_2^4}{R_1^4 - R_2^4} r^{-2}\right). \tag{1.48}$$

1.4 Relevant formulas

1.4.1 Fourier orthogonality

The following orthogonality properties of the sine/cosine functions are the basis of Fourier analysis:

$$\frac{1}{L}\int_0^{2L} \sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{2L}) = \delta_{mn}, \quad n,m \in \mathbb{N}^+ \tag{1.49}$$

$$\frac{1}{L}\int_0^{2L} \cos(\frac{n\pi x}{L})\cos(\frac{m\pi x}{2L}) = \frac{\delta_{mn}}{2^{\delta_{m,0}}}, \quad n,m \in \mathbb{N}, \tag{1.50}$$

$$\frac{1}{L}\int_0^{2L} \cos(\frac{n\pi x}{L})\sin(\frac{m\pi x}{2L}) = 0. \tag{1.51}$$

In some types of applications we often need the following relations:

$$\frac{2}{L}\int_0^{L} \sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{2L}) = \delta_{mn}, \quad n,m \in \mathbb{N}^+, \tag{1.52}$$

$$\frac{2}{L}\int_0^{L} \cos(\frac{n\pi x}{L})\cos(\frac{m\pi x}{2L}) = \frac{\delta_{mn}}{2^{\delta_{m,0}}} \quad n,m \in \mathbb{N},, \tag{1.53}$$

which are a simple consequence of the two above (notice that instead (1.51) above does not generalize to the half-interval).

In problems with mixed Dirichlet-Neumann boundary conditions, the following are useful:

$$\frac{2}{L}\int_0^{L} \sin(\frac{(n+\frac{1}{2})\pi x}{L})\sin(\frac{(n+\frac{1}{2})\pi x}{2L}) = \delta_{mn}, \quad n,m \in \mathbb{N}, \tag{1.54}$$

$$\frac{2}{L}\int_0^{L} \cos(\frac{(n+\frac{1}{2})\pi x}{L})\cos(\frac{(n+\frac{1}{2})\pi x}{2L}) = \delta_{mn}, \quad n,m \in \mathbb{N}, \tag{1.55}$$

The standard Fourier decomposition of a function defined on an interval $[0, 2L]$ involves both sine and cosine functions.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)\right). \tag{1.56}$$

[‡]We could have started already with an ansatz with only the relevant angular part, and then fixed the last two coefficients

this defines a function of period $2L$. The coefficients can be found explicitly using orthogonality. For instance, integrating $\sin(\frac{n\pi x}{L})$ against the function, and using the orthogonality above, we deduce $b_n = \frac{1}{L}\int_0^{2L} dx\ \sin(\frac{n\pi x}{L}) f(x)$.

Often in PDE problems we want to use a different type of series expansion, which is needed to capture the boundary conditions. Typically, we use a basis of trigonometric functions which corresponds to the standard Fourier decomposition of a *larger* interval.

For instance, consider a function $f(x)$ defined on the interval $x \in [0, L]$. In problems where we want to impose Dirichlet-Dirichlet boundary conditions, we use an expansion as a *sine-series*:

$$f(x) = \sum_{n=1}^{\infty} d_n \sin(n\pi x/L). \tag{1.57}$$

Notice that this defines a function of period $2L$, which is odd: $f(x) = -f(-x)$. It is the *odd extension* of the original function. The coefficients are given by $d_n = \frac{2}{L}\int_0^{L} dx\ \sin(\frac{n\pi x}{L}) f(x)$, as can be deduced using the orthogonality relations (1.52) above.

1.4.2 Radial coordinates

Radial coordinates in 2D The Laplace operator in $2D$ is $\Delta^{(2D)} \equiv \partial_x^2 + \partial_y^2$. In radial coordinates it becomes:

$$\partial_r^2 + \frac{\partial_r}{r} + \frac{\partial_\theta^2}{r^2}, \tag{1.58}$$

where the radial coordinates are $r = \sqrt{x^2 + y^2}$, $\theta = \arccos\frac{x}{r} = \arcsin\frac{y}{r}$.

Notable radial equation in 2D - Laplace The ODE

$$R''(r) + \frac{R'(r)}{r} - \frac{m^2 R(r)}{r^2} = 0 \tag{1.59}$$

arises studying Laplace equation $\Delta u = 0$ in radial coordinates in 2D.

This equation has two independent solutions: $R(r) = r^{\pm m}$ for $m > 0$, and for $m = 0$ the two independent solutions are: $R(r) = \text{const}$, $R(r) = \log r$.

Notable radial equation in 2D - Helmoltz The ODE

$$R''(r) + \frac{R'(r)}{r} + \frac{\lambda\, r^2\, R(r) - m^2 R(r)}{r^2} = 0 \tag{1.60}$$

arises studying Helmoltz equation $\Delta u = -\lambda u$ in radial coordinates in 2D.

It can be transformed into Bessel equation for $y(x) = R(\sqrt{\lambda}x)$: $y''(x) + \frac{y'(x)}{x} + \frac{x^2 - m^2}{x^2} y(x) = 0$.

Assume $\text{Re}(m) \geq 0$. Then the solution of Bessel equation with behaviour x^m for $x \sim 1$ is called *Bessel function of the first kind* $J_m(x)$. It has infinitely many zeros on the positive real axis (including $x = 0$ if $m > 0$).

Bessel functions satisfy the following orthogonality relations:

$$\int_0^1 x J_\alpha(\mu_{\alpha,k}x) J_\alpha(\mu_{\alpha,l}x) \propto \delta_{kl}, \tag{1.61}$$

$\forall \alpha$, $k = 1, 2, \ldots$, where $\mu_{\alpha,k}$ denote the zeros, $J_\alpha(\mu_{\alpha,k}) = 0$, $k = 1, 2, \ldots$.

Radial coordinates in 3D The Laplace operator in $3D$ is $\Delta^{(3D)} \equiv \partial_x^2 + \partial_y^2 + \partial_z^2$. In radial coordinates it becomes:

$$\partial_r^2 + \frac{2\partial_r}{r} + \frac{\partial_\theta^2 + \cot\theta \partial_\theta + \frac{\partial_\phi^2}{\sin^2\theta}}{r^2}, \tag{1.62}$$

where the radial coordinates are $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \arccos \frac{z}{r}$, $\phi \arcsin \frac{y}{r \sin\theta} = \arccos \frac{x}{r \sin\theta}$.

Notable radial equation in 3D - Laplace The ODE

$$R''(r) + \frac{2R'(r)}{r} - \frac{l(l+1)R(r)}{r^2} = 0, \tag{1.63}$$

arises studying Laplace equation $\Delta u = 0$ in radial coordinates in 3D.

This equation has two independent solutions: $R(r) = r^l$ and r^{-l-1}.

Notable radial equation in 3D - Helmoltz Studying the more general Helmoltz equation, $\Delta^{(3D)} u = -\lambda u$, the radial part leads to the ODE

$$R''(r) + \frac{2R'(r)}{r} - \frac{l(l+1)R(r) - \lambda r^2 R(r)}{r^2} = 0, \tag{1.64}$$

This equation can also be mapped to Bessel equation doing the substitution $y(x) = R(\sqrt{\lambda}x)/\sqrt{x}$. **check it!**

Assume $\mathrm{Re}(l) \geq 0$. The solution with behaviour $R(r) \sim r^l$ can be written as $R(r) \propto \frac{J_{l+\frac{1}{2}}(\sqrt{\lambda}r)}{\sqrt{\lambda r}}$.

Note: The simplest case is for $l = 0$ (which is relevant to decompose the angle-independent part of solutions of Helmoltz's equation on the sphere). In this case, the eigenfunction is simple because $J_{\frac{1}{2}}(x)/\sqrt{x} = \sqrt{\frac{2}{\pi}} \frac{\sin x}{x}$.