1 Exercises on 1st order PDEs and Burgers equation

NOTE: a typo corrected before (1.56).

1.1 Burgers equation and generalisations

Ex. 1 . A slightly different Burgers.

Note: The first 3 questions in this problem can be answered using the material in Lecture 10, while the last question 4) requires the material in Lecture 20.

Consider the generalised inviscid Burgers equation:

$$u_t + u^2 u_x = 0. \tag{1.1}$$

1) Write a solution for the equation with the method of characteristics.

2) Do the solutions to this equation develop gradient singularities with time evolution?

   Hint: Write the solution in implicit form, as was done in the lecture for the inviscid Burgers equation, and starting from that formula deduce an expression for u_x in terms of the initial data. Study if this expression blows up.

3) If the answer to the question above is yes, compute the time it takes for a singularity to form, given the initial profile $u(x, t = 0) = e^{-x^2}$. Compute the location where this first singularity occurs in the (x, t) plane.

4) Consider the equation above as a conservation law:

$$\partial_t(u) + \partial_x(\frac{u^3}{3}) = 0. \tag{1.2}$$

   Consider the discontinuous initial condition: $u(x, t = 0) = \Theta(-x)$, where $\Theta(x)$ is the step function. How does this initial condition evolve, if we want to preserve the integral form of the conservation law?

   Hint: Discontinuous solutions to conservation laws evolve according to the (appropriate version) of the Rankine-Hugoniot constraints discussed in the lectures. In the case of a step function, the solution preserves its shape but moves with a speed given by the Rankine-Hugoniot conditions. Use the appropriate form of this condition to compute the speed of the shock front.

Solution

1) Using the method of characteristics, we deduce

$$u(x, t) = u_0(s), \tag{1.3}$$

   where $u(s, 0) = u_0(s)$ is the initial condition, and

$$x = s + (u_0(s))^2 t. \tag{1.4}$$

In implicit form, we can write the solution as

$$u = u_0(x - u^2 t). \tag{1.5}$$

2) A solution to this equation will generally produce gradient singularities. In fact, from the implicit solution we get

$$u_x = u_0'(s)\,(1 - 2uu_x t) \longrightarrow u_x = \frac{u_0(s)}{1 + 2u_0(s)u_0'(s)t}. \tag{1.6}$$

The vanishing of the denominator to this equation produces the gradient singularity. It can only vanish after a certain amount of time, if the initial data are smooth, namely

$$t_{\text{shock}} = \min\left\{\left(-\frac{1}{2u_0(s)u_0'(s)}\right), \quad \text{for } s \text{ such that } -\frac{1}{2u_0(s)u_0'(s)} \geq 0\right\}. \tag{1.7}$$

3) With the initial profile $u_0(s) = e^{-s^2}$, we get

$$t_{\text{shock}} = \min\left\{\left(\frac{e^{2s^2}}{4s}\right), \text{ for } s \geq 0\right\}. \tag{1.8}$$

The minimum is reached at $s = \frac{1}{2}$ and gives

$$t_{\text{shock}} = \frac{e^{\frac{1}{2}}}{2}, \quad \longrightarrow x_{\text{shock}} = \frac{1}{2} + (u_0(\frac{1}{2}))^2 t_{\text{shock}} = \frac{1}{2} + \frac{1}{2} = 1. \tag{1.9}$$

4) If we want to preserve the conservation law

$$\partial_t(u) + \partial_x(\frac{u^3}{3}) = 0, \tag{1.10}$$

a moving discontinuity must satisfy the Rankine-Hugoniot equation

$$v_{\text{shock}} = \frac{\frac{u_L^3}{3} - \frac{u_R^3}{3}}{u_L - u_R} = \frac{1}{3}\,(u_L^2 + u_R^2 + u_L u_R), \tag{1.11}$$

with u_L, u_R the values of the discontinuous solutions on the left and right side of the jump. If we have a discontinuous initial condition: $u(x, t = 0) = \Theta(-x)$, with $u_L = 1$, $u_R = 0$, the discontinuity will move uniformly with velocity given by the formula above:

$$v_{\text{shock}} = \frac{1}{3}. \tag{1.12}$$

Thus the solution will be

$$u(x, t) = \Theta(-x + t/3). \tag{1.13}$$

Notice that it is only because of the extremely simple form of the initial condition (which is flat, apart for the discontinuity) that we have v_{shock} independent on time. In general, we would need to still use the method of characteristics, and introduce the discontinuity at the appropriate place using the Rankine-Hugoniot equations (which give an additional differential equation for the position of the shock).

1.2 Initial value problems and general solutions for linear equations

Instructions: Solve the following initial value problems, or state if they do not admit a solution and why. Write also a fully general solution (NOTE: this can be done since these 1st order PDE's are linear - using the method in Lecture 9).

Hint: To check that the initial condition problem can have a solution, check that the Cauchy curve is not parallel to the characteristics.

Ex. 2

Solve:

$$u_x + yu_y = 2u, \tag{1.14}$$

with initial condition $u(1, s) = s$, $s \in \mathbb{R}$.

Find a formula for the general solution of this PDE and check the solution you found to the above IVP is correct.

Solution We look for characteristics

$$\frac{dy}{dx} = y \longrightarrow y = e^x C \tag{1.15}$$

Change variables we choose $\xi = x$ and $\eta = ye^{-x}$ so that η is constant along characteristics.

Thus we get

$$u_\xi = 2u \longrightarrow u = e^{2x} F(ye^{-x}) \tag{1.16}$$

with an arbitrary function F. To match the initial condition we have $F(s) = s/e$, so we have

$$u(x, y) = ye^{x-1}. \tag{1.17}$$

Ex. 3

Solve:

$$u_x + u_y = u^2, \tag{1.18}$$

with initial condition $u(s, 0) = s^2$, $s \in \mathbb{R}$.

Find also a formula for the general solution.

Solution You can proceed in the same way as before. The special solution with these initial conditions is

$$u = \frac{(x-y)^2}{1 - y(x-y)^2}, \tag{1.19}$$

while the general solution is

$$u = \frac{1}{y - F(x-y)}, \tag{1.20}$$

for an arbitrary function F.

Ex. 4

Solve:

$$xu_x + (y + x^2)u_y = u, \tag{1.21}$$

with initial condition $u(2, s) = s - 4$, $s \in \mathbb{R}$.

Find also a formula for the general solution.

Solution: For the special solution with this boundary condition you should find $u(x, y) = y - x^2$. For the general solution you should find $u(x, y) = xF\left(\frac{y-x^2}{x}\right)$, with F an arbitrary function.

Ex. 5 . In 3 variables:

- Solve (if possible) the initial value problems:

$$zu_x + yzu_z = 0 \tag{1.22}$$

  with initial condition $u(x, y, 1) = x^y$.

- Consider the different equation:

$$zu_x + yzu_y = 0 \tag{1.23}$$

  with the same initial condition $u(x, y, 1) = x^y$.

If the problem does not have a solution, explain why.

Solution

- The equation for characteristics is (with t the parameter along characteristic curves):

$$\partial_t x = z, \quad \partial_t y = 0, \quad \partial_t z = zy, \quad \partial_t u = 0. \tag{1.24}$$

  The solution with initial conditions $(x, y, z)|_{t=0} = (x_0, y_0, 1)$, $u_0 = x_0^{y_0}$ is

$$(x, y, z) = \left(x_0 + \frac{1}{y_0}e^{y_0 t}, y_0, e^{y_0 t}\right), \quad u = x_0^{y_0}, \tag{1.25}$$

  which can be written in terms of the explicit variables as:

$$u(x, y, z) = \left(x - \frac{z}{y}\right)^y. \tag{1.26}$$

- This second problem is not well-posed, and does not have solution. In fact, the characteristics in this case lie on the initial Cauchy surface and create a contradiction with the initial condition.

We can see this formally if we compute the determinant (below, x_0, y_0 parametrize the Cauchy surface $(x_0, y_0, 1)$, t is the parameter along characteristics):

$$\begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial y}{\partial x_0} & \frac{\partial z}{\partial x_0} \\ \frac{\partial x}{\partial y_0} & \frac{\partial y}{\partial y_0} & \frac{\partial z}{\partial y_0} \\ \frac{\partial x}{\partial t}\Big|_{t=0} & \frac{\partial y}{\partial t}\Big|_{t=0} & \frac{\partial z}{\partial t}\Big|_{t=0} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & y_0 & 0 \end{vmatrix} = 0. \tag{1.27}$$

1.3 More complicated equations: quasi-linear and fully nonlinear

Hint: Use the method of characteristics, see Lectures 9-11 (if needed, use the version of the method for fully nonlinear equations). For all these problems, check that the initial condition allows for a solution, i.e., the Cauchy curve is not parallel to the characteristics.

Ex. 6

Solve:

$$u_x u_y = 2, \tag{1.28}$$

with initial condition $u(s, s) = 3s$, $s \in [0, 1]$.

Solution Using the method for the completely nonlinear case, we get the characteristic equations (using t for the parameter along the characteristics, and the symbols $p \equiv u_x$, $q \equiv u_y$):

$$\partial_t x = q, \quad \partial_t y = p, \quad \partial_t p = \partial_t q = 0, \quad \partial_t u = 4. \tag{1.29}$$

We need to find the initial conditions on the initial curve parametrised by s. We have:

$$x(s,t)|_{t=0} = s, \quad y(s,t)|_{t=0} = s, \quad u(s,t)|_{t=0} = 3s, \tag{1.30}$$

for $s \in [0, 1]$. To find the initial conditions for p, q we use the constraints:

$$\text{PDE: } pq = 2, \quad \text{and } \partial_s u|_{t=0} = 3 = (u_x \partial_s x + u_y \partial_s y)|_{t=0} = (p+q)|_{t=0}\,. \tag{1.31}$$

This has two independent solutions:

$$\text{case A: } p(s, t=0) = 2, \quad q(s, t=0) = 1, \quad \text{or:} \tag{1.32}$$

$$\text{case B: } q(s, t=0) = 2, \quad p(s, t=0) = 1. \tag{1.33}$$

We present the solution for both.

- Case A: in this case, the solution of the characteristics with initial conditions gives:

$$p(s,t) = 2, \quad q(s,t) = 1, \tag{1.34}$$

$$x(s,t) = s + 2t, \quad y(s,t) = s + t, \tag{1.35}$$

$$u(s,t) = 3s + 4t. \tag{1.36}$$

  for $s \in [0, 1]$. The solution turns out to be simply $u = x + 2y$, in the range $0 \le s \le 1 \longrightarrow 0 \le 2y - x \le 1$ (the initial data do not specify the solution outside).

- Case B: in this case, we can obtain the solution simply by exchanging $x \leftrightarrow y$, so $u = y+2x$, for $0 \leq 2x - y \leq 1$.

Notice that the solution in this case turns out to be linear, but this is just a coincidence of the simple initial data we gave. A generic solution would be complicated.

Ex. 7

Solve:

$$u_x u_y = u, \tag{1.37}$$

with initial condition $u(s,1) = s$, $s \in [0,1]$.

(Leave the solution in implicit form as obtained with the method of characteristics).

Solution: It can be done with the method of the previous exercise. In this case we find for the initial condition in p, q:

$$p(s, t=0) = 1, \quad q(s, t=0) = s, \tag{1.38}$$

and:

$$x(s,t)|_{t=0} = s, \quad y(s,t)|_{t=0} = 1, \quad u(s,t)|_{t=0} = s. \tag{1.39}$$

The characteristics read:

$$\partial_t x = q, \quad \partial_t y = p, \quad \partial_t p = p, \quad \partial_t q = q, \quad \partial_t u = 2u. \tag{1.40}$$

with solution

$$p(s,t) = e^t, \quad q(s,t) = se^t, \tag{1.41}$$
$$x(s,t) = se^t, \quad y(s,t) = e^t, \tag{1.42}$$
$$u(s,t) = se^{2t}. \tag{1.43}$$

Inverting s, t for x, y we find:

$$u(x,y) = xy, \tag{1.44}$$

for $0 \leq x/y \leq 1$.

Ex. 8

Solve (if possible) the following initial value problems for the PDE:

$$uu_x + u^2 u_y = u, \tag{1.45}$$

1) With initial condition: $u(s,1) = s$, $s \in [0,1]$.

2) With initial condition: $u(s,s) = 1$, $s \in [0,1]$.

Note: One of these problems does not admit a solution: which one and why?

Solution It is not necessary but we can divide by u the whole equation. We can solve it also in the original form.

$$u_x + uu_y = 1. \tag{1.46}$$

1) With initial condition: $u(s,1) = s$, $s \in [0,1]$.

   Using the method of characteristics:

$$\partial_t x = 1, \quad \partial_t y = u, \quad \partial_t u = 1. \tag{1.47}$$

   Imposing the initial conditions we solve and find:

$$x(s,t) = s + t, \;\; y(s,t) = 1 + st + \frac{t^2}{2}, \;\; u(s,t) = s + t. \tag{1.48}$$

   So in this particular simple case:

$$u(x,y) = x. \tag{1.49}$$

2) With initial condition: $u(s,s) = 1$, $s \in [0,1]$.

   This second problem **does not admit a solution**. This is because the characteristics lie along the initial curve (s,s). We can see that by computing the determinant

$$\left| \begin{array}{cc} \partial_t x & \partial_t y \\ \partial_s x & \partial_s y \end{array} \right|_{t=0} = \left| \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right| = 0. \tag{1.50}$$

   We would also see that the characteristic system cannot be solved with those initial conditions.

Ex. 9

Solve:

$$u_t + uu_x = u, \tag{1.51}$$

with initial condition $u(x, t = 0) = x^2$, $x \in [0,1]$. Leave the solution in implicit form with the method of characteristics.

Solution This can be solved with the same method as above, the solution with this initial condition is described by:

$$x(s,t) = s^2(e^t - 1), \quad u = s^2 e^t, \quad s \in [0,1], \tag{1.52}$$

where t, the parameter along the characteristics, coincides with the variable t in the original PDE.

Ex. 10 - Initial value problem, both specific and abstract

Consider the PDE:

$$u_x + \frac{1}{2}u_y^2 = 1, \tag{1.53}$$

- Find the solution with the method of characteristics, corresponding to the initial condition $u(0, y) = y^2$. In this case, you should be able to write $u(x, y)$ explicitly.
- Write a solution with the method of characteristics, corresponding to the initial condition $u(0, y) = U_0(y)$. Does the solution ever become multi-valued? Specify if this depends on $U_0(y)$ and how.
- Show that u_x and u_y do *not* blow up for the solution, for any initial condition $U_0(y)$.

Hint: For the first point, the answer is $u(x, y) = x + \frac{y^2}{1+2x}$.

Observation: Notice that the solution can become multi-valued even without u_x or u_y blowing up. In the present case, what happens is that u_{yy} blows up, and after this u_y, rather than u, becomes discontinuous.

Solution

- It can be solved with the method for fully nonlinear equations (also detailed below). The solution is $u(x, y) = x + \frac{y^2}{1+2x}$.
- The equations for characteristics give (setting $p \equiv u_x$, $q \equiv u_y$):

$$\partial_t x = 1, \quad \partial_t y = q, \quad \partial_t p = \partial_t q = 0, \quad \partial_t u = p + q^2 = 2 - p, \tag{1.54}$$

where we used the PDE: $p + q^2/2 = 1$ in the last step.

The initial condition is

$$x_(s, t = 0) = 0, \quad y(s, t = 0) = s, \quad u(s, t = 0) = U_0(s), \tag{1.55}$$

and using the PDE and the constraint $u_s(s, t = 0) = U_0'(s) = (p\partial_s x + q\partial_s y)|_{t=0}$ (which becomes $U_0'(s) = q(s, 0)$, we get

$$q(s, 0) = U_0'(s), \quad p(s, 0) = 1 - \frac{(U_0'(s))^2}{2}. \tag{1.56}$$

The solution of the characteristics sytem is:

$$x(s, t) = t, \quad y(s, t) = s + U_0'(s)t, \tag{1.57}$$

$$q(s, t) = U_0'(s), \quad p(s, t) = 1 - \frac{(U_0'(s))^2}{2}, \tag{1.58}$$

$$u(s, t) = U_0(s) + t + t\frac{(U_0'(s))^2}{2}. \tag{1.59}$$

This is the solution of the PDE in implicit form. Provided we can locally invert $(s,t) \to (x,y)$, we can express u as function of x, y.

The solution becomes multi-valued if the characteristics cross. The condition for the crossing of characteristics is that the map $(s,t) \to (x,y)$ is not invertible. To check this we must compute the Jacobian:

$$\begin{vmatrix} \partial_s x & \partial_s y \\ \partial_t x & \partial_t y \end{vmatrix} = \begin{vmatrix} 0 & 1 + tU_0''(s) \\ 1 & U_0'(s) \end{vmatrix} = -(1 + tU_0''(s)). \tag{1.60}$$

So the condition for this system to produce a multivalued solution is that, for some value of t, and s, the term $1 + tU_0''(s)$ vanishes. Notice that this is similar to the condition for Burgers' equation, but now there is the second derivative of the initial condition involved, rather than the first.

Given the data of the initial condition we can compute the earliest characteristic time when the first singularity (i.e., multi-valuedness) occurs. This is given by

$$t_{\text{sing}} = \min \left\{ -\frac{1}{U_0''(s)}, \text{ for } s \text{ such that } -\frac{1}{U_0''(s)} \geq 0 \right\}. \tag{1.61}$$

This was not asked in the exercise, but in class we started* studying the concrete example where $U_0(s) = e^{-s^2}$. In this case from the condition above one would find that the singularity forms for a value corresponding to $s = 0$, where $-\frac{1}{U_0''(s)}$ has a local positive minimum (its only positive miminum), correspondingly we have $t_{\text{sing}} = -\frac{1}{U_0''(s)}\Big|_{s=0} = 1/2$.

- The last question is on the type of singularity. In this case, as we have shown, there is multi-valuedness. However, contrary to Burgers' equation, for the present PDE u_x does not blow up (instead u_{xx} does). We can see immediately that u_x does not blow up at the singularity, because $u_x = p$ and from the solution for $p(s,t)$ we see that nothing bad happens when $1 + tU_0''(s) = 0$ or for any other values of s, t, provided the initial profile $U_0(s)$ is regular.

Ex. 11 - Eikonal equation with variable propagation speed

Consider light propagating in a 2D material such that light travels with speed $c(x,y) = y$ proportional to the distance from the x-axis. In the geometric optics approximation, the phase of a wave with unit frequency travelling in the medium satisfies the eikonal equation:

$$u_x^2 + u_y^2 = \frac{1}{y^2}, \tag{1.62}$$

Consider the evolution of the initial wave front $u(x = 0, y) = 0$, $y > 0$ (i.e. the positive y-axis). What is the shape of the other wave fronts (i.e., level curves of u) in the (x,y) plane?

*In class I made some mistakes in this calculation, in the very last minutes of the recording, when considering this concrete case. Here is explained what would have been the correct statements.

What is the path travelled by light rays in the medium?

Hint: As explained in the lecture, this is just the path traced by the characteristic curves for this equation.

Solution The characteristics system with the usual notation is:

$$\partial_t x = 2p, \quad \partial_t y = 2q, \quad \partial_t p = 0, \quad \partial_t q = -2/y^3, \quad \partial_t u = 2/y^2. \tag{1.63}$$

with IC:

$$x(s, t = 0) = 0, \quad y(s, t = 0) = s, \quad u(s, t = 0) = 0, \tag{1.64}$$

and combining the PDE: $p^2 + q^2 = 1/y^2$ with $\partial_s u(s, t = 0) = 0 = u_x \partial_s x + u_y \partial_s y|_{t=0} = q(s, 0)$, we get

$$q(s, 0) = 0, \quad p(s, 0) = \pm 1/s. \tag{1.65}$$

We can easily integrate the equations for p and x.

$$p(s, t) = \pm 1/s, \quad x(s, t) = \pm 2t/s. \tag{1.66}$$

Dividing the equations for y and q for each other we get:

$$\frac{dy}{dq} = -qy^3 \longrightarrow y^{-2} = q^2 + \text{const} \tag{1.67}$$

and imposing the initial condition we fix the constant to $y^{-2} = q^2 + s^{-2}$, or $q^2 = y^{-2} - s^{-2}$. Plugging this expression into the equation for y, we get $\partial_t y = 2\sqrt{y^{-2} - s^{-2}}$, which can be solved (with our initial condition) to

$$y(s, t) = s\sqrt{1 - 4\frac{t^2}{s^4}}. \tag{1.68}$$

Notice that $x(s, t)$ and $y(s, t)$ satisfy:

$$x(s, t)^2 + y(s, t)^2 = s^2, \tag{1.69}$$

therefore this answers the last question: rays travel along characteristic curves, which are circumferences crossing the imaginary axis orthogonally.

We can already answer the first equation also, geometrically: the wave fronts are perpendicular to the characteristics, therefore they must be rays of the form $y/x = \tan\theta$ for fixed θ the angle of the ray with the x-axis.

To write the solution completely and check that indeed u is constant on these rays, we can solve the last characteristic ODE for u. This gives (imposing $u = 0$ on the y-axis, which is our initial condition):

$$\partial_t u = \frac{2}{s^2 - 4\frac{t^2}{s^2}} \longrightarrow u(s, t) = \text{arctanh}\left(2\frac{t}{s^2}\right) = \pm\text{arctanh}\left(\frac{x}{\sqrt{x^2 + y^2}}\right). \tag{1.70}$$

Indeed the value of u (apart for the discrete choice $\pm$) depends only on $\frac{x}{\sqrt{x^2+y^2}} = \cos(\theta)$, and is clearly constant on the rays.