Example of exam test. English version

The threshold to be admitted to the oral exam is 17/32 points.
Duration: 2h
You can use the file "UsefulEquations.pdf" (which will be distributed in class during the exam), in addition you can use your own collection of notes prepared at home (2 pages maximum).*

Exercise 1. (10 points)

A quantity $u(x,t)$ is described by the wave equation on the interval $x \in [0, 2\pi]$:

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in [0, 2\pi],\ t \geq 0,$$

with boundary conditions

$$u_x(0,t) = u_x(2\pi, t) = 0.$$

Write the solution for times $t \geq 0$, satisfying the initial condition $u(x, t=0) = \sin^2(\frac{x}{2})$ and $\partial_t u(x,t)|_{t=0} = 0$ for $x \in [0, 2\pi]$.
NOTE: Fourier coefficients should be defined explicitly as integrals, but it is NOT necessary to evaluate the integrals.

Exercise 2. (10 points)

Consider the following differential equation for the field $u(x,t)$:

$$u_t + (u + x)u_x = 0,$$

with the initial condition $u(x,0) = 4/(1+x^2)$, $x \in \mathbb{R}$.

- (9 points) Express the solution using the method of characteristics. The solution can be left in implicit form.
- (1 point) Show that the solution develops a singularity with time evolution. Motivate your answer with an equation describing the time when the singularity is formed. (It is not necessary to solve the equation and compute the time explicitly).

Exercise 3. (6 points)

Consider the differential equation:

$$\frac{8y(x)}{9(x+1)(x-2)(x-3)^2} + \left(\frac{3}{2(x+1)} + \frac{1}{2(x-2)}\right) y'(x) + y''(x) = 0, \tag{0.1}$$

for a function $y(x)$. Using the Papperitz-Riemann method, write a basis of two independent solutions. These solutions should be written explicitly in terms of special functions, and should

*Two sides, not 4 sides!

be distinguished by their different behaviour in a neighbourhood of one of the three Fuchsian points (you can choose which one).

NOTE: You can use as information that the equation has indices $(\frac{1}{3}, \frac{2}{3})$ relative to the Fuchsian singular point $x = 3$, and $(0, -\frac{1}{2})$ relative to the Fuchsian singular point $x = -1$.

Exercise 4. (6 points)

Consider a two-dimensional metal plate shaped like a sector of a circle, described in polar coordinates by the domain $0 \leq r \leq 2$, $0 \leq \theta \leq \frac{3}{5}\pi$.

The plate has a temperature distribution† $T(r, \theta)$ which is **time-independent**.

On the boundaries of the plate the temperature is kept at the following values:

$$T(r = 2, \theta) = \sin(\frac{5}{3}\theta) + 2\sin(5\theta), \quad 0 \leq \theta \leq \frac{3}{5}\pi, \tag{0.2}$$

$$T(r, \theta = 0) = 0, \quad T(r, \theta = \frac{3}{5}\pi) = 0, \quad 0 \leq r \leq 2. \tag{0.3}$$

Write the temperature distribution $T(r, \theta)$ inside the plate. NOTE: there is no time dependence.

Suggested strategy: Decompose the relevant differential equation in polar coordinates. Decomposing $T(r, \theta) = \sum_n c_n \Theta_n(\theta) R_n(r)$, find the correct basis of eigenfunctions by imposing the boundary conditions on the straight edges of the boundary. Finally, fix the coefficients of the decomposition by imposing the boundary condition on the curvilinear part of the boundary.

†Ignore heat transfer in the direction orthogonal to the plate. Assume that the system is strictly two-dimensional.