Exercises on ODEs part 2

NOTE: typo corrected in (0.22).

0.1 Series expansions solutions

Exercises

Ex 1. Discuss the possible singularities of the ODE

$$x^4 y'' + y = 0, \tag{0.1}$$

and in particular the form of the solution at $x \sim 0$ and $x \sim \infty$.

Ex. 2 - Classification of singular points. Study the singular points of the following equations and discuss what form the solution can take around these points, and the radius of convergence of the possible expansions.

a)

$$x^2 y'' + xy' + (x^2 - a^2)y = 0.$$

(Bessel equation) In this case, compute the form of the series expansion around $x = 0$. (This was done in class).

b)

$$(1 - x^2)y'' - 2xy' + a(a+1)y = 0.$$

(Legendre equation).

c)

$$xy'' + (1 + a - x)y' + by = 0$$

(Laguerre equation),

where a, $b \in \mathbb{C}$.

Ex. 3 Study the equation:

$$x^2(x-2)y'' + xy' - y = 0.$$

Notice that it has only 3 singularities, all Fuchsian. The singularities are: $0, 2, \infty$, with indices $(1, \frac{1}{2})$, $(0, \frac{1}{2})$, $(0, -1)$, respectively. Use the P-symbol method to write a basis of solutions around $x = 2$.

Ex 4. Consider the example of this ODE:

$$x(x+1)y'' - (x-1)y' + y = 0 \tag{0.2}$$

- Discuss the possible singular points, and their type.
- Find explicitly the first two terms of two independent series solutions around $x = 0$.
- Write a solution around $x = 0$ with the P-symbol method and check the above result.

Solutions

Ex. 1 $x = 0$ is an irregular singularity, so there is no series solution around $x = 0$ with finitely many negative terms . The solution will have an essential singularity and a Laurent series with infinitely many negative powers.

$x = \infty$ is a regular singular point. In fact, writing the equation as $y'' + p(x)y + q(x) = 0$, $q(x) \sim O(1/x^4)$, and $p(x) \sim 0/x$ at infinity. Since $0 \neq 2$, infinity is a Fuchsian singularity and not a regular point.

Plugging in the equation $x^{-\rho}$, for $x \to \infty$ we find $\rho(\rho + 1) = 0$. So the indices at infinity are $\{0, -1\}$ and the form of the solutions around infinity are

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{-n}, \quad y_2(x) = A \log(x) y_1(x) + x \sum_{n=0}^{\infty} b_n x^{-n}, \tag{0.3}$$

with A a constant to be fixed. To determine A we must plug the solution into the ODE at large x. We notice that $y_2(x) \sim A \log(x)(1 + a_1/x + a_2/x^2 + \dots) + b_0 x + b_1 + b_2/x + \dots$. Plugging this expansion into the ODE and matching orders at $x \to \infty$, we find that we must have $A = 0$. So we find simply $y_2(x) = x \sum_{n=0}^{\infty} b_n x^{-n}$, where we can assume $b_0 = 1$, $b_1 = 0$, and the other coefficients are fixed by recursion.

Ex. 2 (a) - Bessel equation In this case there are two singular points (for generic parameter a): $x = 0$ is a Fuchsian singularity, and $x = \infty$ is an irregular singularity.

The solution cannot be found around infinity with the series expansion method.

We can write the solution as a series of the form:

$$y(x) = x^{\rho} \sum_{n=0}^{\infty} c_n x^n, \tag{0.4}$$

Plugging this expansion in the ODE we find at leading order:

$$\left[(\rho - 1)\rho + \rho - a^2\right] c_0 = 0, \tag{0.5}$$

so $\rho = \pm a$.

The following orders give in general:

$$c_n \left[(a + n)(a + n - 1) + (a + n) - a^2\right] + c_{n-2} = 0, \tag{0.6}$$

where $c_{-1} = c_{-2} = 0$. For $n = 0$, this is automatically satisfied. For $n = 1$, it gives $c_1 = 0$, the next orders give

$$c_n = -c_{n-2} \frac{1}{n(n + 2a)}, \quad n \geq 2. \tag{0.7}$$

This implies that all odd coefficients are zero (since $c_1 = 0$):

$$c_{2k+1} = 0. \tag{0.8}$$

For the even ones, we can iterate the previous equation to lower the index until we find:

$$c_{2k} = (-1)^k c_0 \frac{1}{(2k)!!\,[(2k+2a)(2k+2a-2)\dots(2a+2)]}. \tag{0.9}$$

Noting that $(2k)!! \equiv (2k)(2k-2)\dots 4\cdot 2 = 2^k(k!)$, and that

$$(2k+2a)(2k+2a-2)\dots(2a+2) \ = 2^k\,(a+1)(a+2)\dots(a+k) = (a+1)_k = \frac{\Gamma(a+1+k)}{\Gamma(a+1)},$$

then we can write the solution as

$$\begin{aligned} y(x) &= x^a \sum_{k=0} c_{2k} x^{2k} = x^a c_0 \sum_{k=0} (-1)^k \left(\frac{x}{2}\right)^{2k} \frac{1}{(k!)(a+1)_k} && (0.10)\\ &= x^a c_0 \sum_{k=0} (-1)^k \left(\frac{x}{2}\right)^{2k} \frac{\Gamma(a+1)}{(k!)\Gamma(a+1+k)}. && (0.11)\end{aligned}$$

This solution (normalised with $c_0 = \frac{1}{\Gamma(a+1)2^a}$) is denoted as

$$J_a(x) \equiv \sum_{k=0} (-1)^k \left(\frac{x}{2}\right)^{2k+a} \frac{1}{(k!)\Gamma(a+1+k)} \tag{0.12}$$

(Bessel function of the first kind), for generic $a \in \mathbb{C}$. If a is not integer, the two independent solutions as $J_{\pm a}(x)$. If $a \in \mathbb{N}$, then the solution $J_a(x)$ still has the same form. The other solution will in general also contain a log contribution and can be obtained as a limit of the situation with $a \notin \mathbb{N}$.

Notice also that we can recognise the form above as a special kind of generalised hypergeometric:

$$J_a(x) \propto {}_0F_1(\,;a+1;-\frac{x}{2}). \tag{0.13}$$

Ex. 2 (b) - Legendre

- $x = 1$ is a Fuchsian singularity with indices $\rho = \{0, 0\}$. The solution could have the form:

$$y_1(x) = \sum_{n\geq 0} a_n (x-1)^n, \quad y_2(x) = \sum_{n\geq 0} b_n (x-1)^n + A \log(x-1)\, y_1(x). \tag{0.14}$$

- $x = -1$ is a Fuchsian singularity with indices $\rho = 0, 1$. The solution could have the form:

$$y_1(x) = \sum_{n\geq 0} a_n (x+1)^n, \quad y_2(x) = \sum_{n\geq 0} b_n (x+1)^n + A \log(x+1)\, y_1(x). \tag{0.15}$$

- $x = \infty$ is a Fuchsian singularity with indices $\rho = a+1, -a$. The solution could have the form:

$$y_1(x) = x^{-a-1} \sum_{n\geq 0} a_n x^{-n}, \quad y_2(x) = x^a \sum_{n\geq 0} b_n x^{-n}. \tag{0.16}$$

Ex. 2 (c) - Laguerre

- $x = \infty$ is an irregular singularity.
- $x = 0$ is a Fuchsian singularity with indices $\rho = 0, -a$. So the solution could have the form:
$$y_1(x) = \sum_{n \geq 0} a_n x^n, \quad y_2(x) = x^{-a} \sum_{n \geq 0} b_n x^n. \tag{0.17}$$

Ex. 3 The solution can be represented in the P-symbol notation as

$$y(x) = P \left\{ \begin{array}{cccc} & 2 & 0 & \infty \\ x & 0 & 1 & -1 \\ & \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right\}. \tag{0.18}$$

Since we are interested in behaviour around $x = 2$, first we map the points $z_1 = 2$, $z_2 = 0$, $z_3 = \infty$ to the canonical positions 0,1,∞. This is done with the fractional linear transformation:

$$x \to \frac{2 - x}{2}, \tag{0.19}$$

so we have

$$y(x) = P \left\{ \begin{array}{cccc} & 0 & 1 & \infty \\ \frac{2-x}{2}, & 0 & 1 & -1 \\ & \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right\}. \tag{0.20}$$

Now we want to bring it to canonical form, i.e. we must have one zero index in column 1 and one in column 2. We are not there yet. So we use another property to redefine the indices:

$$y(x) = \left(\frac{2 - x}{2} - 1 \right) P \left\{ \begin{array}{cccc} & 0 & 1 & \infty \\ \frac{2-x}{2}, & 0 & 0 & 0 \\ & \frac{1}{2} & -\frac{1}{2} & 1 \end{array} \right\}. \tag{0.21}$$

(Notice how we wanted to change the indices for the point 1, but they get automatically redefined also at infinity!)

This is now in canonical form, so from here we can read one solution:*†

$$y_1(x) = \left(-\frac{x}{2} \right) {}_2F_1(0, 1; 1/2; 1 - \frac{x}{2}) = -\frac{x}{2}. \tag{0.22}$$

We denoted this as y_1 since it has the leading behaviour at $x \sim 2$.

The second independent solution with the leading behaviour $(x - 2)^{\frac{1}{2}}$ is found by starting with the equation written as (i.e., we swap the two indices for z_1).

$$y(x) = P \left\{ \begin{array}{cccc} & 0 & 1 & \infty \\ \frac{2-x}{2}, & \frac{1}{2} & 1 & -1 \\ & 0 & \frac{1}{2} & 0 \end{array} \right\}. \tag{0.23}$$

*Check the relevant formula in the notes or in the "UsefulEquations" file.

†(Note: actually in this case, since $(0)_i = \delta_{i,0}$, the hypergeometric reduces to 1 and the solution is simply $\propto x$. This is just a coincidence of the data of the problem).

Then, using the property to redefine the indices:

$$y(x) = \left(\frac{2-x}{2}\right)^{\frac{1}{2}} \left(\frac{2-x}{2} - 1\right) P \left\{ \begin{array}{cccc} & 0 & 1 & \infty \\ \frac{2-x}{2}, & 0 & 0 & \frac{1}{2} \\ & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{array} \right\}. \tag{0.24}$$

From this canonical form we read the other independent solution:‡

$$y_2(x) = \left(\frac{2-x}{2}\right)^{\frac{1}{2}} \left(\frac{2-x}{2} - 1\right) {}_2F_1(\frac{1}{2}, \frac{3}{2}; \frac{3}{2}; \frac{2-x}{2}). \tag{0.25}$$

Ex. 4

- The singularities are $z_1 \equiv 0$, $z_2 \equiv -1$, $z_3 \equiv \infty$. We can check that they are all Fuchsian.

  The behaviour of solutions around z_1 $(x = 0)$ is $y \sim x^\alpha$, plugging in the ODE we find the indicial equation $\alpha^2 = 0$.

  Solutions around z_2 $(x = -1)$ behave like $y \sim (x+1)^\alpha$, plugging in the ODE we find the indicial equation $\alpha(\alpha - 2) = 0$ (which is again a resonant case since they differ by an integer).

  Around z_3 (infinity), taking the form $y \sim x^{-\alpha}$ and expanding the ODE for $x \to \infty$ we find $\alpha^2 + 2\alpha + 1 = 0$, so the indices are both -1.

- Around $x = 0$, since the indices are 0 and 0 (degenerate case), we can take two solutions of the form: $y_1(x) = a_0 + a_1 x + \dots$ and $y_2(x) = (b_0 + b_1 x + \dots) + \log(x) A (a_0 + a_1 x + \dots)$. We can normalize $a_0 = 1$ without loss of generality.

  Plugging these expansions in the ODE, at the leading order from the equation for y_1 we find $a_0 + a_1 = 0$, so we can take $y_1(x) = 1 - x + O(x^2)$.

  From the equation for $y_2(x)$ we then find, at leading order, $4A - b_0 - b_1 = 0$. Subtracting a quantity proportional to y_1, we can assume without loss of generality $b_0 = 0$. Therefore, we find $y_2(x) = A \log x \left(1 - x + O(x^2)\right) + 4Ax + O(x^2)$ (we can set $A = 1$ for normalisation).

- With the P-symbol notation the generic solution is written as

$$y(x) = P \left\{ \begin{array}{cccc} & 0 & -1 & \infty \\ x & 0 & 0 & -1 \\ & 0 & 2 & -1 \end{array} \right\} \tag{0.26}$$

‡In this case it turns out this is a simple algebraic function (again, a coincidence of the data of the problem which produce a "simple" hypergeometric function).

(Notice the subtlety that the solution behaves like $\sim x^1$ at $x \sim \infty$, but this corresponds to index -1, not $+1$, because the "distance" from infinity is $1/x$, not x).

Using the fractional linear transformation $x \to x' = -x$, we rewrite this in the canonical form:

$$y(x) = P \left\{ \begin{array}{cccc} & 0 & 1 & \infty \\ -x & 0 & 0 & -1 \\ & 0 & 2 & -1 \end{array} \right\}. \tag{0.27}$$

This is in canonical form with $c = 1$, $a = b = -1$, so we can read one solution immediately:

$$y(x) =_2 F_1(-1, -1; 1; -x). \tag{0.28}$$

This is given by a power series around $x = 0$, starting as $1 - x + \dots$, so indeed it agrees with one of the solutions above.[§]

It is a bit trickier to describe the second solution with the P-symbol method because of the degeneracy, although it can also be done by introducing regularising parameters.

[§]Note: in fact, since $a \in \mathbb{Z}_{<0}$, the solution truncates and it is just a polynomial $y_1(x) = 1 - x$.