# Complements of Mathematical Methods for Physics, written exam 18/01/2023

*The threshold to be admitted to the oral exam is 17/32 points.*
*Duration: 2h*
*You can use your own collection of notes prepared at home (2 pages* maximum).*

**Exercise 1. (10 points)**

Consider the heat equation for the temperature along a one-dimensional bar of length 3:

$$u_t - \alpha u_{xx} = 0, \quad x \in [0,3], \quad t \geq 0,$$

with the initial condition:

$$u(x, t=0) = x^2(3-x)^2 + 1, \quad x \in [0,3],$$

and with the endpoints kept at constant temperature $T = 1$:

$$\begin{aligned} u(x=0,t) &= 1, \quad t \geq 0, \\ u(x=3,t) &= 1, \quad t \geq 0. \end{aligned}$$

Compute the time evolution of the solution for $t \geq 0$ using the Fourier method. Fourier coefficients should be expressed explicitly as integrals, but it is NOT necessary to evaluate the integrals.

**Suggestion:** Divide $u(x,t) = u_0 + v(x,t)$, where $u_0$ is a constant and $v(x,t)$ satisfies homogeneous boundary conditions. Compute $v(x,t)$ with the Fourier method.

**Exercise 2. (10 points)**

Find the **general solution** of the following partial differential equation for the function $u(x,y)$:

$$u_x + ((1+y)\cos x)\ u_y = 1.$$

*Two sides, not 4 sides!

**Exercise 3. (6 points)**

Consider the differential equation:

$$y''(x) + \left(\frac{4}{3(x-2)} + \frac{1}{2(x-4)}\right) y'(x) + \left(-\frac{11}{36(x-2)} + \frac{11}{36(x-4)} - \frac{1}{2(x-4)^2}\right) y(x) = 0.$$

Using the Papperitz-Riemann method, find the particular solution of the equation which is regular for $x \to 2$ and normalized as

$$\lim_{x\to 2} y(x) = A,$$

with $A$ a constant. The solution should be expressed explicitly in terms of special functions.
**Note:** To shorten the calculations, you can use the information that the indices of the equation in the Fuchsian singular point $x = 2$ are $\left\{0, -\frac{1}{3}\right\}$, while the indices in the Fuchsian singular point $x = \infty$ are $\left\{\frac{2}{3}, \frac{1}{6}\right\}$.

**Suggested steps:**

- Compute the indices in the remaining Fuchsian point;
- write the generic solution in terms of the P-symbol;
- using appropriate transformations, bring the P-symbol to a canonical form which is convenient to extract explicitly the solution of interest.
- Normalize the solution appropriately.

**Exercise 4. (6 points)**

Consider a square drum. Let us assume that the transverse displacement of the membrane, denoted as $u(x, y, t)$ for $(x, y) \in [0, L] \times [0, L]$ and time $t \in \mathbb{R}$, satisfies the wave equation with propagation speed $c$.

Taking the initial condition:

$$\begin{aligned} u(x, y, t = 0) &= \sin\left(\frac{\pi}{L}x\right) \times \sin\left(\frac{3\pi}{L}y\right), \\ \partial_t u(x, y, t)\big|_{t=0} &= \sin\left(\frac{4\pi}{L}x\right) \times \sin\left(\frac{\pi}{L}y\right), \end{aligned}$$

compute the evolution of $u(x, y, t)$ for $t \geq 0$.