BURGERS' EQUATION PART 2

IN THE PART OF THE LECTURES ON 1st ORDER PDE's, WE HAVE PREVIOUSLY STUDIED THE INVISCID BURGERS PDE:

$$u_t + u u_x = 0$$

WE SHOWED THAT SOLUTIONS FORM GRADIENT SINGULARITIES, AFTER WHICH THE CHARACTERISTIC CURVES CROSS.
NOW WE DISCUSS SOME MORE ASPECTS OF THIS EQUATION: 1. HOW TO MAKE SENSE OF SOLUTIONS AFTER THE TIME OF THE SINGULARITY, AND 2. THE EXTENSION OF THE EQUATION WITH VISCOSITY:

$$u_t + u u_x - \underbrace{\mu}_{\text{VISCOSITY}} u_{xx} = 0$$

(ALSO JUST CALLED "BURGERS EQUATION")

LET US START BY RECALLING THAT:

INVISCID BURGERS $\approx$ TOY MODEL FOR EULER EQUATIONS

FULL BURGERS EQUATION $\approx$ TOY MODEL FOR NAVIER-STOKES

IN FACT WE WILL SEE THAT THERE IS AN IMPORTANT PROPERTY THAT IS CORRECTLY CAPTURED BY THE

TOY MODEL!

THE EULER EQUATIONS PRODUCE GRADIENT SINGULARITIES JUST LIKE INVISCID BURGERS.

THIS PHENOMENON IS IN FACT VERY GENERAL FOR NONLINEAR SYSTEMS OF FIRST-ORDER CONSERVATION LAWS.

THE NAVIER-STOKES ARE BELIEVED TO HAVE INSTEAD SOLUTIONS THAT REMAIN SMOOTH: VISCOSITY REGULARIZES SINGULARITIES. THIS CAN BE SHOWN RIGOROUSLY FOR THE VISCOUS BURGERS EQUATION, AS WE WILL SEE.

(FOR THE 3D NAVIER-STOKES EQUATIONS, PROVING THIS RESULT IS STILL AN OPEN PROBLEM! IN FACT, IT IS ONE OF THE "MILLENNIUM PROBLEMS" IN MATHEMATICS!).

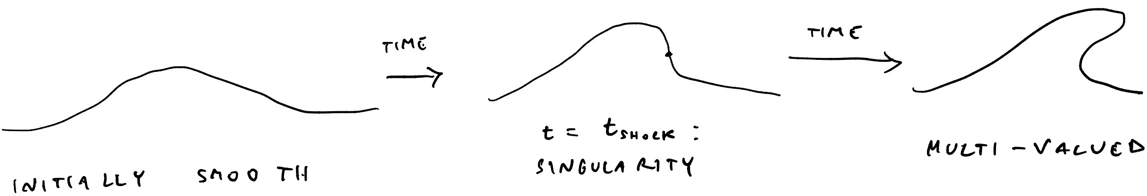
* OF COURSE, IN SOME ASPECTS THE 1D VERSION OF BURGERS EQUATION IS MUCH SIMPLER THAN THE HIGHER-DIMENSIONAL EXAMPLES.

ONE ASPECT IS TURBULENCE WHICH IS EXPECTED TO BE PRODUCED BY N-S IN 3D BUT IS ABSENT IN THE BURGERS EQUATION.

BEFORE DISCUSSING THE CASE WITH VISCOSITY, LET US CONSIDER A BIT MORE THE SIMPLEST CASE:

$$u_t + u\,u_x = 0.$$

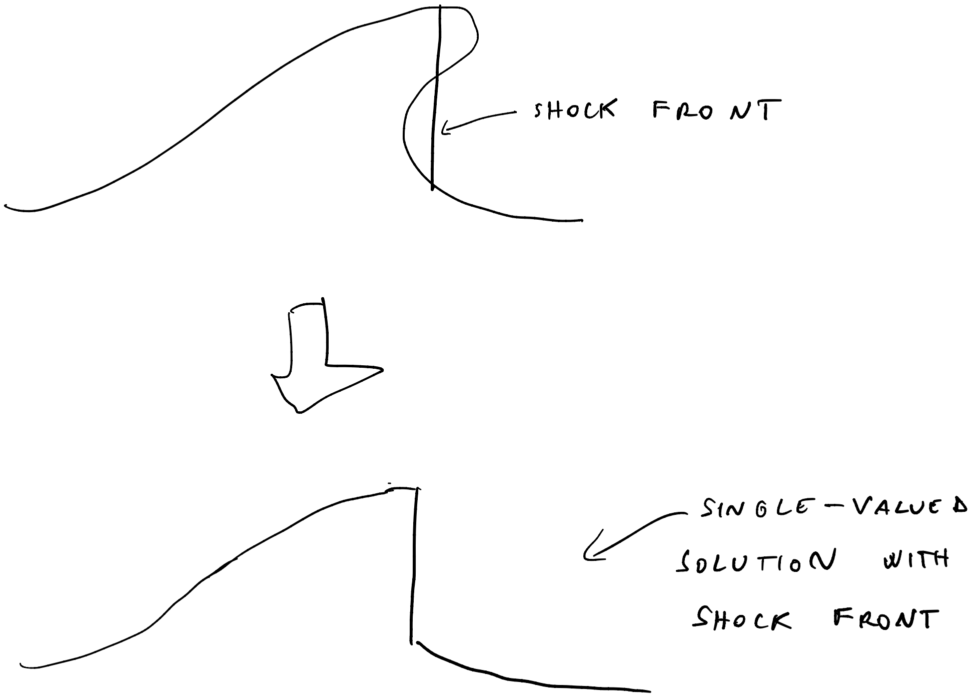
IN MOST APPLICATIONS, u SHOULD BE SINGLE-VALUED. CAN WE STILL DO SOMETHING WITH OUR MODEL AFTER THE SOLUTION BREAKS DOWN?



LET US ASSUME THAT THE MODEL FAILS WHEN THE GRADIENT u_x BECOMES TOO LARGE. SO WE SHOULD NOT TRUST THE SOLUTION WHERE IT ACTUALLY BREAKS AND BECOMES MULTI-VALUED. BUT LET US ASSUME THAT WE CAN STILL TRUST THE SOLUTION, EVEN WHEN $t > t_{shock}$, IN THE PARTS WHERE u_x IS NOT LARGE.

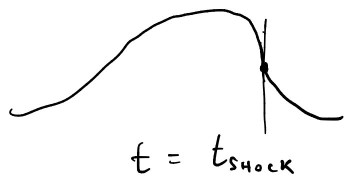
WHAT WE WANT TO DO IS TO RESTORE SINGLE-VALUEDNESS BY INTRODUCING A DISCONTINUITY:

↓



THIS IS CALLED A SHOCK, OR SHOCK FRONT.

IT IS FORMED AFTER THE TIME WHEN FIRST $u_x \to \infty$,



, AND PERSISTS FOR ALL THE FOLLOWING TIMES.

FAR FROM THE SHOCK, THE SOLUTION FOLLOWS THE STANDARD BURGERS EQ., AND u_x IS NEVER TOO LARGE.

THE SHOCK PHYSICALLY IS A REPLACEMENT FOR

A REGION OF VERY LARGE $|u_x|$, WHERE WE ASSUME THAT THE ORIGINAL BURGERS EQ. IS NOT VALID ANYMORE. IN REALITY, THE SHOCK IS REPLACED BY SOME VERY THIN LAYER, AND IN FACT VERY THIN SHOCK LAYERS ARE OBSERVED ALSO EXPERIMENTALLY. (SO IT IS NOT JUST A TRICK, IT DESCRIBES PHYSICS).

HOWEVER WE STILL NEED TO SOLVE TWO ISSUES:

- MATHEMATICALLY HOW DO WE MAKE SENSE OF SOLUTIONS WITH DISCONTINUITIES?

- IS THE POSITION OF THE SHOCK ARBITRARY? (IF YES, THEN WE ARE NOT PREDICTING ANYTHING USEFUL...)

THE ANSWER TO THESE QUESTIONS ARE RELATED. IN FACT, IF WE WANT TO MAKE MATHEMATICAL SENSE OF THE DISCONTINUOUS SOLUTIONS, WE MUST REFORMULATE IT IN A WAY THAT ALSO TELLS US WHERE THE SHOCK SHOUD BE - AND HOW IT MOVES. THE DYNAMICS OF THE

SHOCK IS FIXED (SO WE ARE PREDICTIVE!).

THE KEY IS USING THE PHYSICAL MEANING OF THE EQUATION AS A CONSERVATION LAW.

IN FACT, NOTICE THAT THIS PDE HAS THE FORM

$$\rho_t + j_x = 0$$

WITH $\rho = u$, $j = \frac{u^2}{2}$.

OR: $\partial_t (u) + \partial_x \left(\frac{u^2}{2} \right) = 0$

THIS IS A CONSERVATION LAW: IT IS SIMPLE TO SEE BY INTEGRATING THIS LAW THAT IT IMPLIES

$$\frac{d}{dt}\left(\int_{M_1}^{M_2} \rho(x', t)\, dx' \right) = - \left[j(M_2, t) - j(M_1, t) \right]$$

FOR ANY M_1, M_2

INTEGRAL FORM OF THE CONSERVATION LAW

NOTE: THIS IS JUST FOUND (ASSUMING A STANDARD, DIFFERENTIABLE SOLUTION) AS FOLLOWS:

$$\frac{d}{dt}\int_{M_1}^{M_2} \rho(x',t)\,dx' = \int_{M_1}^{M_2} \partial_t \rho(x',t)\,dx'$$

USING $\rho_t + j_x = 0$

$$\stackrel{\downarrow}{=} -\int_{M_1}^{M_2} \partial_{x'} j(x',t)\,dx'$$

$$= j(M_1,t) - j(M_2,t).$$

Comment: the more rigorous and complete mathematical definition of discontinuous solutions to conservation laws (which we did not discuss) can be found in Stavroulakis and Tersian's book. It goes under the name of "weak solutions".

IN OUR CONCRETE CASE:

$$\boxed{\frac{d}{dt}\left(\int_{M_1}^{M_2} u(x',t)\, dx' \right) = -\frac{1}{2}\left(u^2(M_2,t) - u^2(M_1,t) \right) \quad \forall\ M_1, M_2 \in \mathbb{R}.}$$

WE WANT TO USE THE INTEGRATED CONSERVATION LAW AS A REFORMULATION OF THE EQUATION, VALID ALSO FOR PIECEWISE DISCONTINUOUS SOLUTIONS.

(IN FACT, THERE IS NO PROBLEM APPLYING THE INTEGRATED FORM TO PIECEWISE NON-DIFFERENTIABLE OR DISCONTINUOUS FUNCTIONS!)

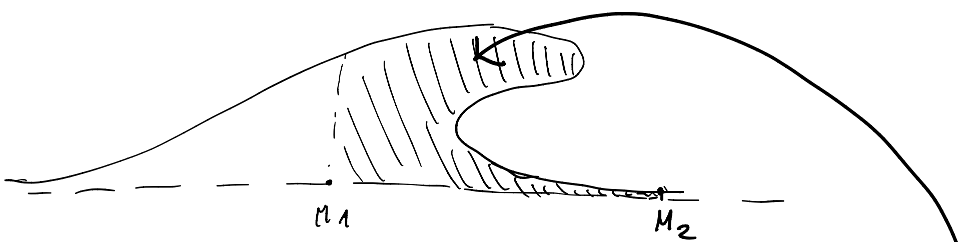
DEMAND: THE POSITION OF THE SHOCK SHOULD BE FIXED SUCH THAT THE INTEGRATED CONSERVATION LAW IS STILL VALID.

IN THIS WAY WE KEEP INTACT THE MEANING OF THE EQUATION AS A CONSERVATION LAW.

WHAT DOES IT MEAN?

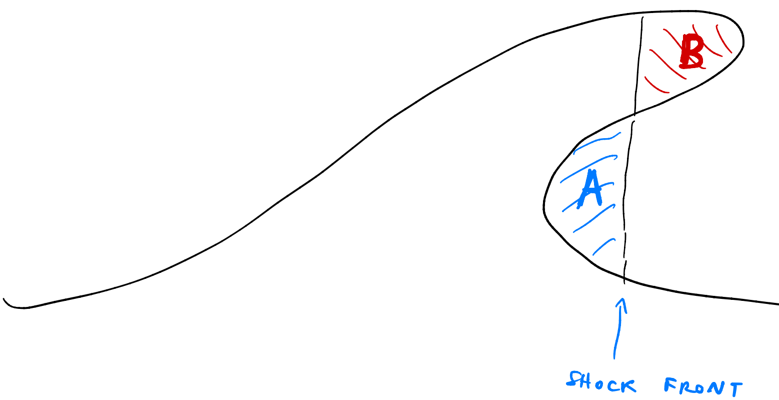
NOTICE THAT THE INTEGRATED CONSERVATION LAW IS ALSO VALID FOR THE MULTI VALUED SOLUTION, PROVIDED IN THE MULTIVALUED PARTS WE DO THE INTEGRALS PIECE BY PIECE.

SO FOR INSTANCE IN THIS CASE:



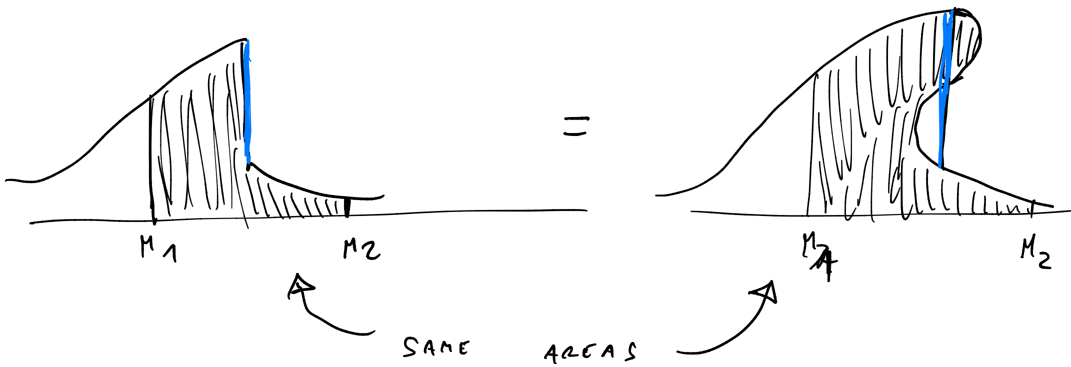
$$\frac{d}{dt}\int_{M_1}^{M_2} u(x',t)\,dx' = j(M_1)-j(M_2),$$ WHERE $\int_{M_1}^{M_2} u \; dx'$ MEANS THE INTEGRAL BETWEEN THE REAL AXIS AND THE CURVE

IF WE WANT TO MANTAIN THE CONS. LAW AFTER INTRODUCING A SHOCK, IT MEANS THAT THE AREA UNDER THE CURVE SHOULD REMAIN THE SAME!



A AND B SHOULD HAVE EQUAL AREAS!

IN THIS WAY, FOR ALL M_1, M_2 BEFORE AND AFTER THE SHOCK, WE HAVE:



THIS IS THE DEFINITION IN PICTURES OF WHERE THE SHOCK SHOULD BE (IT CAN ALSO BE TURNED INTO EQUATIONS).

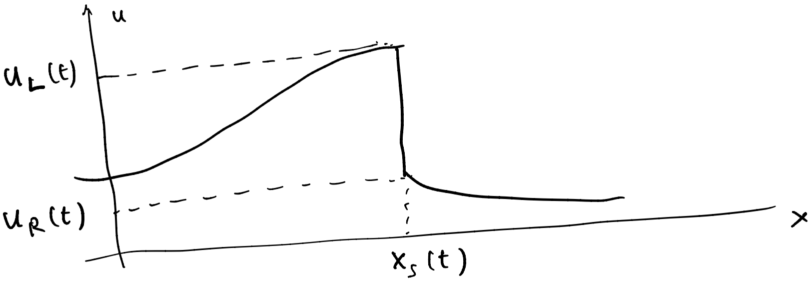
LET US DEDUCE FROM THIS AN IMPORTANT PROPERTY: A RELATION ON THE SPEED WITH WHICH THE SHOCK FRONT MOVES.

TO OBTAIN IT, WE CONSIDER A DISCONTINUOUS SOLUTION, AND IMPOSE THE CONSERVATION LAW IN INTEGRATED FORM.

LET: $X_s(t)$ = POSITION OF THE SHOCK FRONT

$u_L(t)$ = VALUE OF u IMMEDIATELY LEFT OF THE SHOCK

$u_R(t)$ = VALUE OF u IMMEDIATELY RIGHT OF THE SHOCK.



TAKE $M_1 < x_s(t) < M_2$

WE WANT TO IMPOSE :

$$\underbrace{\frac{d}{dt}\int_{M_1}^{M_2} u(x',t)\,dx'}_{\text{LHS}} = \underbrace{\frac{1}{2}\left[u^2(M_1,t) - u^2(M_2,t)\right]}_{\text{RHS}}$$

LET US WRITE THE FIRST INTEGRAL AS

$$\int_{M_1}^{M_2} = \int_{M_1}^{x_s(t)} + \int_{x_s(t)}^{M_2} .$$

THEN :

$$\frac{d}{dt}\int_{M_1}^{M_2} u(x,t)\,dx = \frac{d}{dt}\left[\int_{M_1}^{x_s(t)} u\,dx + \int_{x_s(t)}^{M_2} u\,dx\right]$$

$$= \dot{x}_s(t)\cdot\left(u_L(t) - u_R(t)\right)$$

$$+ \int_{M_1}^{x_s(t)} (\partial_t u)\,dx + \int_{x_s(t)}^{M} (\partial_t u)\,dx$$

NOTICE THAT IN THE INTERVALS $x \in (M_1, x_s(t))$,

OR $x \in (x_s(t), M_2)$, u IS DIFFERENTIABLE

AND SATISFIES THE PDE, SO UNDER THE TWO

INTEGRALS WE CAN REPLACE $u_t = -j_x$

AND SO WE FIND:

$$\int_{M_1}^{x_s(t)} \partial_t u \, dx = \frac{1}{2}\left(u^2(M_1, t) - u_L^2(t) \right),$$

NOTICE THEY ARE DIFFERENT

$$\int_{x_s(t)}^{M_2} \partial_t u \, dx = \frac{1}{2}\left(u_R^2(t) - u^2(M_2, t) \right),$$

SO WE FIND

$$\text{LHS} = \frac{d}{dt} \int_{M_1}^{M_2} u(x,t)\, dx =$$

$$= \dot{x}_s(t) \left(u_L(t) - u_R(t) \right)$$

$$+ \frac{1}{2} \left(u_R^2(t) - u_L^2(t) \right)$$

$$+ \frac{1}{2} \cdot \left(u^2(M_1, t) - u^2(M_2, t) \right)$$

AND FROM LHS = RHS WE FINALLY GET:

$$\boxed{\dot{x}_s (u_L - u_R) + \frac{1}{2}(u_R^2 - u_L^2) = 0}$$

(WE DROPPED THE DEPENDENCE ON t)

WHICH GIVES AN EQUATION FOR THE SPEED OF THE SHOCK FRONT

$$\boxed{\dot{x}_s = \frac{1}{2}\frac{(u_L^2 - u_R^2)}{u_L - u_R} = \frac{1}{2}(u_L + u_R)}$$

THIS IS CALLED RANKINE–HUGONIOT CONDITION

FOR A GENERIC CONSERVATION LAW, IT WOULD GENERALISE TO:

$$\boxed{\dot{x}_s = \frac{j_L - j_R}{\rho_L - \rho_R}}$$

, IT ALWAYS RELATES THE SHOCK'S SPEED TO THE DISCONTINUITIES OF ρ AND j.

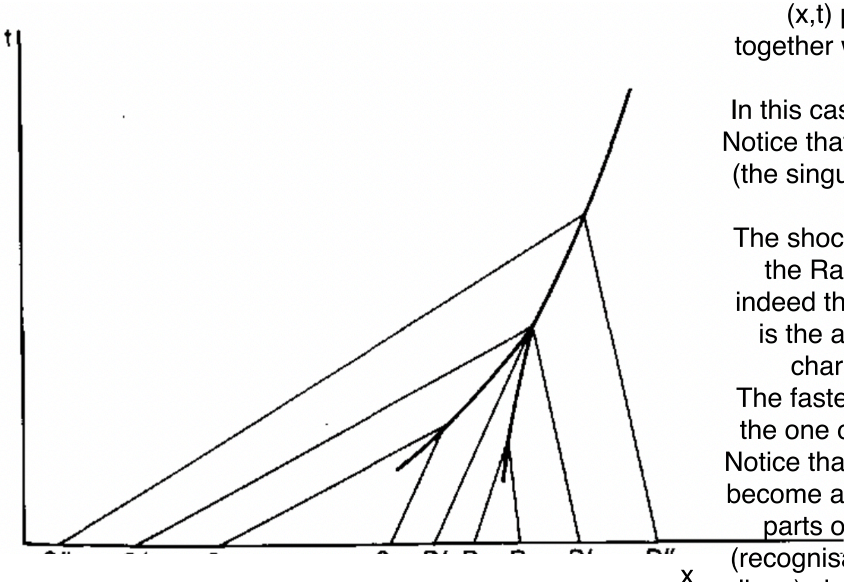
COMMENT: FOR THE BURGERS EQUATION

$$\boxed{\dot{X}_s = \frac{1}{2}\left(u_L + u_R\right)}$$

NOTICE THAT FROM THE SOLUTION WITH CHARACTERISTICS WE KNOW THAT, WHERE THE SOLUTION IS SMOOTH, THE VALUE OF THE FIELD IS TRANSPORTED WITH VELOCITY EQUAL TO u IN EACH POINT.

THE SHOCK MOVES WITH A SPEED GIVEN BY THE AVERAGE OF THE SPEEDS OF POINTS IMMEDIATELY TO ITS LEFT AND RIGHT.

Taken by Whitham “Linear and nonlinear waves”:



The plot shows, for a solution of the inviscid Burgers equation, the shape of characteristic curves in the (x,t) plane (the straight lines), together with the position of the shock fronts (bold lines).

In this case there are two shock fronts. Notice that they appear at a certain time (the singularity time), which is different for each shock.

The shocks move with speed given by the Rankine-Hugoniot equations, indeed the slope of the shocks curves is the average of the slopes of the characteristics on each side.

The faster shock on the left overtakes the one on the right, and they merge.

Notice that over time the shocks tend to become asymptotically slower, and that parts of the solution with higher u (recognisable by steeper characteristic lines) slowly disappear “eaten” by the shocks.

LET US LIST SOME FEATURES OF SOLUTIONS WITH SHOCKS:

- ONCE FORMED, SHOCKS NEVER DISAPPEAR, BUT THEY CAN MERGE WHEN A FASTER SHOCK OVERTAKES A SLOWER ONE.

- IT IS A GENERAL FEATURE THAT, WITH INITIAL DATA WITH $u \to 0$ AT $x \to \pm\infty$, SHOCKS ASYMPTOTICALLY HAVE VANISHING DISCONTINUITIES (AND ALSO THEIR SPEED) AS $t \to \infty$.

EVENTUALLY THE SHOCKS "EAT UP" ALL THE FEATURES OF THE SOLUTION, IN THIS SENSE THEY RESEMBLE A DISSIPATIVE PROCESS, AS FOR $t \to \infty$ THE WHOLE SOLUTION TENDS TO $u \to 0$.

* TECHNICAL COMMENT: NOTICE THAT THERE IS A PRECISE CHOICE WE MADE WHEN WE WROTE THE EQUATION AS

$$\rho_t + j_x = 0$$

WITH $\rho = u$, $j = \frac{u^2}{2}$.

IN FACT, THE INVISCID B. EQUATION ALSO HAS MANY OTHER CONSERVATION LAWS! FOR INSTANCE, MULTIPLYING THE PDE FOR u, WE SEE THAT IT CAN BE REWRITTEN AS:

$$\rho^{(2)}_t + \partial_x j^{(2)} = 0$$

WITH $\rho^{(2)} = u^2$, $j^{(2)} = \frac{2}{3} u^3$,

OR IN MANY OTHER WAYS....

THESE REWRITINGS ARE EQUIVALENT FOR THE PDE (SO THERE IS NO DIFFERENCE FOR SMOOTH SOLUTIONS), BUT THE TWO DIFFERENT REWRITINGS GIVE A DISTINCT DYNAMICS FOR THE SHOCKS!

YOU CAN SEE IT BECAUSE THE RANKINE-HUGONIOT EQUATION DEPENDS ON THE FORM OF ρ AND j. SO, WE MUST CHOOSE WHICH CONSERVATION LAW IS THE PHYSICAL ONE, i.e. THE ONE THAT TELLS THE SHOCKS HOW TO MOVE, AND THE OTHER ONE WILL BE VIOLATED AFTER THE FORMATION OF THE SHOCKS.

IN THE FLUID DYNAMICS APPLICATIONS, THE PHYSICAL CONSERVATION LAW IS THE ONE WITH

$\rho = u$, $j = \frac{u^2}{2}$, AND THE SHOCKS

MOVE AS WE HAVE DESCRIBED ABOVE.

BURGERS EQUATION WITH VISCOSITY

LET US FINALLY SEE WHAT CHANGES IF WE INTRODUCE THE VISCOUS TERM:

$$u_t + u u_x - \mu u_{xx} = 0 \qquad \text{WITH } \mu > 0.$$

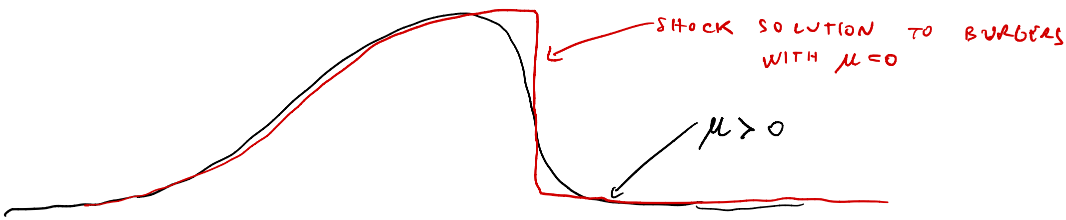
WE WILL SEE THAT:

- THIS EQUATION IS EXACTLY SOLVABLE, THANKS TO A DIRECT LINK WITH THE HEAT EQUATION.

- THE SOLUTION DOES NOT PRODUCE SINGULARITIES AND REMAINS SINGLE-VALUED

- FOR SMALL μ, THE SOLUTION APPROXIMATES ASYMPTOTICALLY THE SOLUTION TO BURGERS' EQUATION, AND, AFTER t_{SHOCK}, $\lim_{\mu \to 0^+} u(x,t)$ BECOMES THE SOLUTION WITH SHOCKS THAT WE HAVE CONSTRUCTED ABOVE.

- FOR FINITE BUT SMALL μ, THE SHOCKS ARE SMOOTHED OUT INTO THIN LAYERS:



LET US DISCUSS HOW TO SOLVE THE EQUATION.

$$u_t + u\, u_x = \mu\, u_{xx}$$

REWRITE AS:

$$u_t = \partial_x \left(-\frac{u^2}{2} + \mu\, u_x \right)$$

WE NOW LOOK FOR ψ SUCH THAT $u = \psi_x$.

↳ THEN THE PDE BECOMES:

$$\psi_{tx} = \partial_x \left(-\frac{u^2}{2} + \mu\, u_x \right) = \partial_x \left(-\frac{\psi_x^2}{2} + \mu\, \psi_{xx} \right)$$

SINCE WE EXPECT $\psi_{tx} = \psi_{xt}$, FROM THE EQ. ABOVE WE GET

$$\boxed{\psi_t = -\frac{\psi_x^2}{2} + \mu \psi_{xx}}$$

THIS DOES NOT SEEM SIMPLER!

HOWEVER IF WE SET $\varphi = e^{-\frac{1}{2\mu}\psi}$

(OR $\psi = -2\mu \log \varphi$)

THEN WE GET:

$$\psi_t = -2\mu \frac{\varphi_t}{\varphi}$$

$$\psi_x = -2\mu \frac{\varphi_x}{\varphi}$$

$$\psi_{xx} = -2\mu \left(\frac{\varphi_{xx}}{\varphi} - \frac{\varphi_x^2}{\varphi^2} \right)$$

AND THE PDE FOR ψ BECOMES

HEAT EQUATION!!

$$\boxed{\varphi_t = \mu \varphi_{xx}}$$

SO THE SOLUTION HAS THE FORM:

$$u = \psi_x = -2\mu \frac{\varphi_x}{\varphi}$$

WITH φ SOLVING THE HEAT EQUATION

$$\varphi_t = \mu \varphi_{xx} .$$

LET US CONSIDER THE CAUCHY PROBLEM ON THE LINE.

INITIAL CONDITION $u(x,0) = u_0(x)$ BECOMES:

$$\log \varphi(x,0) = -\frac{1}{2\mu} \int_0^x u_0(s)\, ds + \underbrace{\cancel{C}}_{\text{ARBITRARY CONSTANT}}$$

(REDEFINE $\varphi \to \text{Const.}\,\varphi$ DOES NOT AFFECT u!) $\to$ SET $C \to 0$

THE HEAT EQUATION HAS AN EXPLICIT SOLUTION (SEE PREVIOUS NOTES!)

$$\varphi(x,t) = \frac{1}{\sqrt{4\pi\mu t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\mu t}} \cdot \varphi(y,0)\, dy$$

$$= \frac{1}{\sqrt{4\pi\mu t}} \int_{-\infty}^{\infty} dy\; e^{-\frac{(x-y)^2}{4\mu t} - \frac{1}{2\mu} \int_0^y u_0(s)\, ds}$$

FROM THIS WE CAN WRITE THE EXPLICIT SOLUTION FOR u:

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{(x-y)}{t} \cdot e^{-\frac{(x-y)^2}{4\mu t} - \frac{1}{2\mu}\int_0^y u_0(s)\,ds} \cdot dy}{\displaystyle\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\mu t} - \frac{1}{2\mu}\int_0^y u_0(s)\,ds}\, dy}$$

* LESSON: WE SEE THAT THE SOLUTION IS SMOOTH AT ALL TIMES BECAUSE OF THE PROPERTIES OF THE HEAT EQUATION! (AND OBVIOUSLY IT REMAINS SINGLE-VALUED)

THE OTHER PROPERTIES WE HAVE LISTED CAN BE ESTABLISHED TAKING THE $\mu \to 0^+$ ASYMPTOTIC LIMIT OF THIS EXPLICIT EXPRESSION. THIS CAN BE DONE USING THE SADDLE-POINT METHOD SEEN FOR STIRLING'S FORMULA.

LET US JUST SCHEMATICALLY DESCRIBE HOW THIS WORKS. WE WILL JUST SHOW HOW THE SOLUTION OBTAINED WITH THE METHOD OF CHARACTERISTICS FOR THE INVISCID B. EQUATION IS REOBTAINED FOR $\mu \sim 0$.

NOTE: WE CANNOT JUST TREAT μ AS A PERTURBATION BECAUSE THIS IS AN EXAMPLE OF "SINGULAR PERTURBATION" (INDEED μ APPEARS IN FRONT OF THE HIGHEST DERIVATIVE TERM).

INSTEAD WE USE THE SADDLE POINT METHOD TO TREAT THE INTEGRALS IN THE EXPLICIT FORMULA.

↳ RECALL THAT FOR INTEGRALS OF THE FORM:

$$I(\mu) = \int dy \; e^{-\frac{S(y)}{2\mu}} \; g(y) \qquad , \quad \mu > 0$$

THE SADDLE-POINT ASYMPTOTIC EXPANSION TELLS US TO EXPAND AROUND THE MINIMUM OF $S(y)$: WE GET

↓

$$\boxed{I(\mu) \sim \sqrt{\frac{4\pi\mu}{S''(y_{min})}}\, g(y_{min})\, e^{-\frac{S(y_{min})}{2\mu}}},$$

FOR $\mu \to 0^+$.

IN OUR CASE IN THE FORMULA FOR u:

$$S(y) = \frac{1}{2t}(x-y)^2 + \int_0^y u_0(s)\, ds$$

AND, FOR THE INTEGRAL IN THE NUMERATOR:

$$g(y) = \frac{(x-y)}{t}$$

WHILE FOR THE INTEGRAL IN THE DENOMINATOR

$$g(y) = 1.$$

SO THE ASYMPTOTIC RESULT FOR u AS $\mu \to 0^+$

$$u \sim \frac{(x - y_{min})}{t},$$

WHERE y_{min} DEFINES THE MINIMUM OF

$$S(y) = \frac{1}{2t}(x-y)^2 + \int_0^y u_0(s)\,ds$$

SO, $S'(y_{min}) = 0$ GIVES:

$$\frac{1}{t}(y_{min} - x) + u_0(y_{min}) = 0$$

$$\hookrightarrow \; y_{min} = x - u_0(y_{min})\,t$$

$\hookrightarrow$ THIS IS THE SAME AS THE EQUATION FOR CHARACTERISTICS

$$x = s + u_0(s)\,t$$

THAT WE FOUND PREVIOUSLY FOR THE INVISCID BURGERS EQ, AND INDEED WE FIND:

$$u \sim \frac{(x - y_{min})}{t} = u_0(y_{min})$$

WITH $x = y_{min} + u_0(y_{min})t$

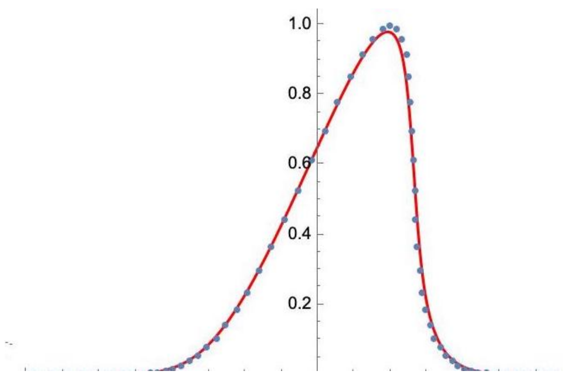
SO WE REOBTAIN THE PREVIOUS RESULT OF THE METHOD OF CHARACTERISTICS. THIS DERIVATION ASSUMES THAT $S(y)$ HAS A SINGLE MINIMUM. WHEN $S(y)$ HAS MORE MINIMA WE ARE IN THE SITUATION WHERE THE CHARACTERISTICS CROSS.

HOWEVER IN THE SADDLE POINT METHOD WE SHOULD ALWAYS PICK THE ABSOLUTE MINIMUM OF $S(y)$ $\longrightarrow$ THUS WE SEE THAT IN THE LIMIT $\mu \to 0^+$ THE SOLUTION REMAINS SINGLE-VALUED AND NATURALLY CHOOSES A "BRANCH".

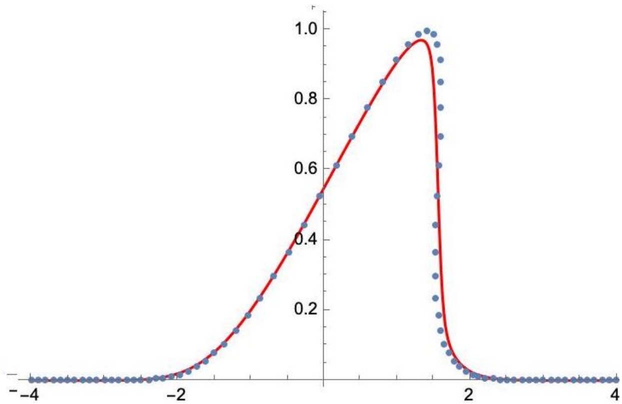
WITH A BIT MORE WORK ONE CAN ALSO SHOW PRECISELY THAT THE $\mu \to 0^+$ ASYMPTOTIC LIMIT REPRODUCES THE SOLUTION WITH SHOCKS, WHERE THE SHOCKS ARE IN THE POSITION GIVEN BY THE RANKINE-HUGONIOT EQUATIONS AND THE CONSERVATION LAW INTERPRETATION STUDIED ABOVE.

THIS IS ILLUSTRATED IN THE FOLLOWING FIGURES.

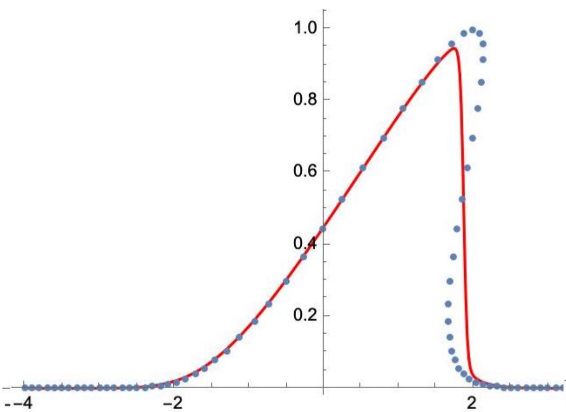
In the plots below we show:
solution of Burgers eq. with viscosity \mu = 0.01 (red)
solution of inviscid Burgers (blue dots)
for initial condition u_0(x) = e^(-x^2).



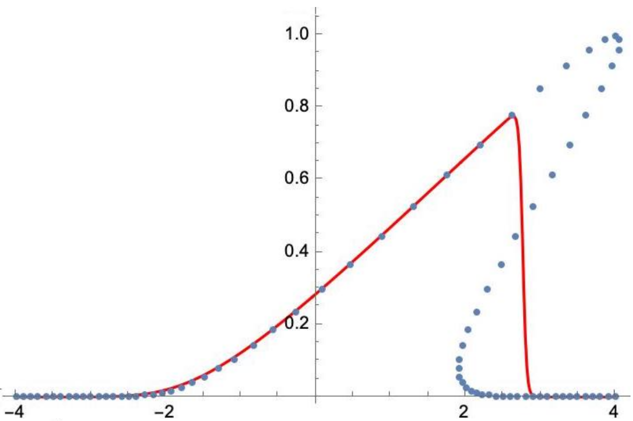
t=1



t=Sqrt[2]
(the singularity time)



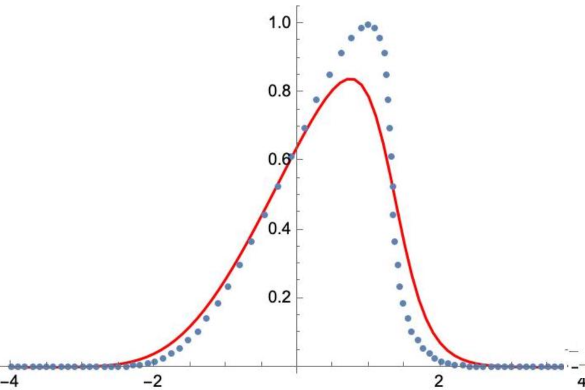
t=2



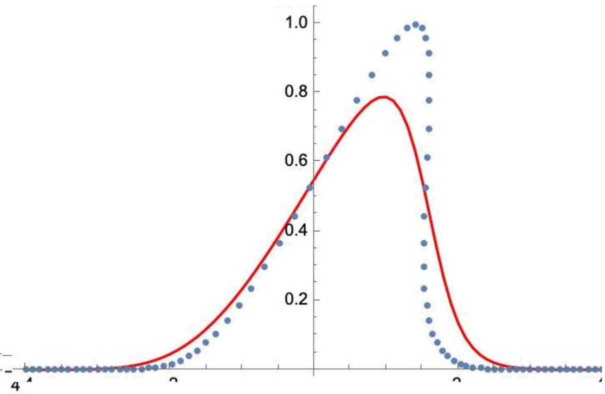
t=4

Notice that the viscous solution (red)
closely approximates a shock front
satisfying the “equal area law”.

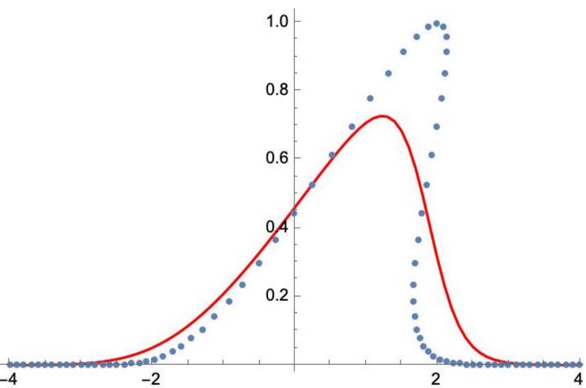
The same as above but with more viscosity: \mu = 0.1.
You can see that the solution with more viscosity approximates less well the inviscid solution, and has a less sharp “shock front”.



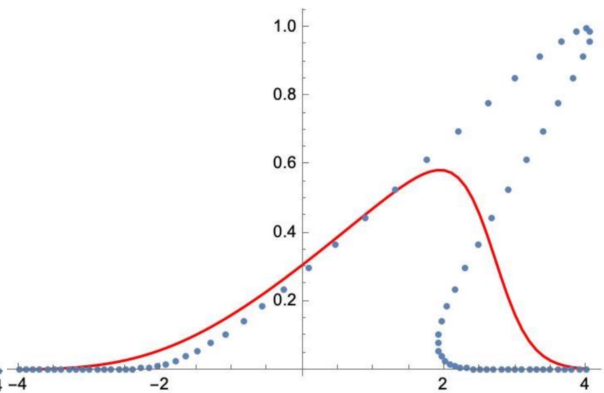
t=1



t=Sqrt[2]



t=2



t=4

IN CLASS, WE ALSO DESCRIBED BRIEFLY ANOTHER IMPORTANT PDE: THE KdV EQUATION

$$u_t + u\,u_x + \underbrace{\mu\, u_{xxx}}_{\text{DISPERSION}} = 0$$

This part was discussed quickly and will not be examined

IN THIS CASE, RATHER THAN DISSIPATION, WE ARE ADDING A DISPERSIVE TERM TO THE INVISCID BURGERS' EQUATION. THE KdV EQUATION HAS A COMPLETELY DIFFERENT BEHAVIOUR: IT IS AN EXAMPLE OF EQUATION WITH DYNAMICS DOMINATED BY "SOLITARY WAVES" OR "SOLITONS", WHICH ARE PULSES WITH A ROBUST SHAPE GIVEN BY A SPECIAL BALANCE BETWEEN NONLINEARITY AND DISPERSION.

REMARKABLY, SOLITONS INTERACT ELASTICALLY WITH EACH OTHER (ALTHOUGH THEY GET A PHASE SHIFT, WHICH IS A NONLINEAR EFFECT) AND THIS IS A HALLMARK OF A VERY SPECIAL MATHEMATICAL PROPERTY (CALLED "INTEGRABILITY") WHICH MAKES THE EQUATION EXACTLY SOLVABLE WITH SOME SPECIAL METHODS.