Answers

Exercise 1.

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(\frac{nx}{2})\, \cos(\frac{ntc}{2}), \tag{0.1}$$

with

$$A_m = \frac{1}{2}\int_0^{2\pi} \sin^2(\frac{x}{2})\cos(\frac{nx}{2}), \quad m = 0, 1, 2, \ldots \tag{0.2}$$

Exercise 2

- Given the initial condition $u(x,0) = u_0(x)$ (in the present case $u_0(x) = \frac{4}{1+x^2}$), the solution is given in implicit form by:

  $$u(x,t) = u_0(s) \tag{0.3}$$

  with

  $$x = e^t s + (e^t - 1)u_0(s) \tag{0.4}$$

- The equation forms a gradient singularity when the Jacobian of the map (x,t) vanishes. The Jacobian is in this case $\partial_s x = e^t + (e^t - 1)u_0'(s)$, so the singularity occurs when $x_s = 0 \longrightarrow e^t = \frac{u'(s)}{1+u_0'(s)}$.

  The time when the singularity forms is described by the earliest time when $x_s = 0$:

  $$t_{\text{shock}} = \min\left\{\log\left(\frac{u'(s)}{1+u_0'(s)}\right), \text{ such that } \log\left(\frac{u_0'(s)}{1+u_0'(s)}\right) > 0\right\}. \tag{0.5}$$

Exercise 3

There are a few different ways to write the solution, depending on the choice of singular point for the expansion and various possible transformations of P-symbols.

Here I give one possible solution, giving a basis of solutions constructed around point $x = -1$. The equation has indices $(0, -\frac{1}{2})$ relative to $x_1 = -1$, $(0, \frac{1}{2})$ relative to $x_2 = 2$, and $(\frac{1}{3}, \frac{2}{3})$ relative to $x_3 = 3$.

With a conformal map we map them to $0, 1, \infty$ and relate the equation to the P-symbol:

$$y = P\left\{\begin{array}{cccc} & 0 & 1 & \infty \\ \frac{(x+1)}{3(3-x)} & 0 & 0 & \frac{1}{3} \\ & -\frac{1}{2} & \frac{1}{2} & \frac{2}{3} \end{array}\right\} \tag{0.6}$$

From this we extract two solutions:

$$y_1 = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; \frac{(x+1)}{3(3-x)}\right) \tag{0.7}$$

and the second independent solution is (obtained with transformations of the P-symbol):

$$y_2 = \left(\frac{(x+1)}{3(3-x)}\right)^{-\frac{1}{2}} {}_2F_1\left(-\frac{1}{6}, \frac{1}{6}; \frac{1}{2}; \frac{(x+1)}{3(3-x)}\right). \tag{0.8}$$

Exercise 4

We should use the polar decomposition of Laplace equation in 2D. The solution is

$$T(r,\theta) = \left(\frac{r}{2}\right)^{\frac{5}{3}} \sin(\frac{5}{3}\theta) + 2\left(\frac{r}{2}\right)^{5} \sin(5\theta). \tag{0.9}$$