COMPLEMENTI DI METODI MATEMATICI

$$\Gamma(z) = \int_0^\infty dt\, e^{-t}\, t^{z-1}, \qquad \mathrm{Re}\, z > 1$$

$$\Gamma(z+1) = z\,\Gamma(z)$$

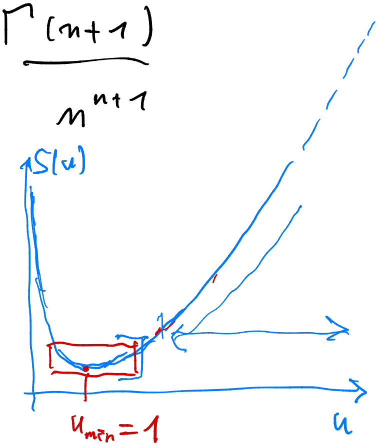
$$\Gamma(n+1) = n!$$

$$\Gamma(n+1) = \int_0^\infty e^{-t}\, t^n\, dt \qquad t = u/n$$

$$= n^{n+1} \cdot \underbrace{\int_0^\infty e^{-un}\, u^n\, du}_{A(n) = \frac{\Gamma(n+1)}{n^{n+1}}}$$

$$A(n) = \int_0^\infty e^{-S(u)\,n}\, du$$

$$S(u) = u - \log u$$



$$S(u) = S(u_{min}) + \frac{S''(u_{min})}{2}(u-u_{min})^2 + \dots$$

$$= 1 + \frac{(u-1)^2}{2} + \dots$$

$$S(1) = 1$$

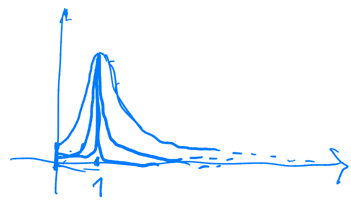
$$S'(u) = 1 - \frac{1}{u}$$

$$S''(u)\Big|_{u_{min}} = \frac{1}{u^2}\Big|_{u_{min}} = 1$$

$$A(n) = \int_0^\infty e^{-S(u)\, n}\, du = \int_0^\infty e^{-n\left(1+\frac{(u-1)^2}{2}+\dots\right)}\, du$$

$$= e^{-n} \int_0^\infty e^{-\frac{(u-1)^2}{2}\cdot n}\, du$$

$$\simeq e^{-n} \int_{-\infty}^{\infty} e^{-\frac{(u-1)^2}{2} n}\, du$$



$$u-1 = \frac{v}{\sqrt{n}} \cdot \sqrt{2}$$

$$\simeq e^{-n} \int_{-\infty}^{\infty} e^{-v^2}\, dv \sqrt{\frac{2}{n}}$$

$$= e^{-n} \sqrt{\frac{2\pi}{n}} \simeq A(n) = \frac{\Gamma(n+1)}{n^{n+1}}$$

$$\Gamma(n+1) \simeq n^n e^{-n} \sqrt{2\pi n}, \qquad n \to \infty$$

$$\log \Gamma(n+1) \simeq n \log n - n + \frac{1}{2}\log(2\pi n)$$

$$+ \sum_{k=2}^{\infty} \frac{B_k}{k(k-1)\, n^{k-1}} \quad , n \to \infty$$

B_k : "NUMERI DI BERNOULLI"

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k\, t^k}{k!}$$

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = B_5 = B_7 = \dots = 0$$

$$B_4 = -\frac{1}{30}, \qquad B_{2k} \neq 0, \quad B_{2k+1} = 0$$

$$k \geq 1$$

$$e^{c_1/n + \frac{c_2}{n^2} + \dots} \qquad n \to \infty$$

$$= 1 + \left(\frac{c_1}{n} + \frac{c_2}{n^2} + \dots\right) + \frac{1}{2!}\left(\frac{c_1}{n} + \frac{c_2}{n^2} + \dots\right)^2 + \dots$$

$$= 1 + \frac{c_1}{n} + \frac{c_2 + \frac{1}{2}c_1^2}{n^2} + \dots$$

$$e^{-n} \sim \sum_{k=0}^{\infty} \frac{c_k}{n^k}, \quad n \to \infty.$$

$$e^{-n} - \sum_{k=0}^{N} \frac{c_k}{n^k} = O\left(\frac{1}{n^{N+1}}\right)$$

$$e^{-n} - c_0 = O\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty}$$

$$\Downarrow$$

$$\lim_{n\to\infty} (e^{-n} - c_0) = 0 \quad \to \quad c_0 = 0$$

$$e^{-n} - \frac{c_1}{n} = O\left(\frac{1}{n^2}\right)$$

$$n e^{-n} - c_1 = O\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} n e^{-n} - c_1 = 0$$

$$c_1 = \lim_{n\to\infty} n e^{-n} = 0$$

$$e^{-n} \sim \sum_{k=1}^{\infty} \frac{0}{n^k}$$

$$B(z_1, z_2) = \int_0^1 dt\; t^{z_1-1} \quad (1-t)^{z_2-1}$$

$$\mathrm{Re}\, z_1 > 1$$

$$\mathrm{Re}\, z_2 > 1$$

$$B(z_1, z_2) = \frac{\Gamma(z_1)\,\Gamma(z_2)}{\Gamma(z_1+z_2)}$$

$$\Gamma(z_1)\,\Gamma(z_2) = \int_0^\infty du\; e^{-u}\; u^{z_1-1} \quad \int_0^\infty dv\; e^{-v}\; v^{z_2-1}$$

$du\, dv$ $\quad\quad$ $u \in (0, \infty)$

$v \in (0, \infty)$

$$u+v = s$$

$$u = s \cdot t \rightarrow t = \frac{u}{u+v}$$

$$v = s \cdot (1-t)$$

$s \in (0, \infty)$

$t \in (0, 1)$

$$du\, dv = ds\, dt \cdot \left| \begin{matrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{matrix} \right|$$

$$= ds\, dt \left| \begin{matrix} t & s \\ 1-t & -s \end{matrix} \right| = ds\, dt \left| \left(-\cancel{t}\, s - s \cdot (1-\cancel{t}) \right) \right|$$

$$ds\, dt\, |-s| = s \cdot ds\, dt$$

$$\Gamma(z_1)\Gamma(z_2) = \int_0^\infty du \int_0^\infty dv\, e^{-u-v}\, u^{z_1-1}\, v^{z_2-1}$$

$$= \int_0^\infty ds \int_0^1 dt\; s \cdot e^{-s}\, s^{z_1-1}\, s^{z_2-1}\, t^{z_1-1}\, (1-t)^{z_2-1}$$

$$= \left(\underbrace{\int_0^\infty ds\, e^{-s}\, s^{z_1+z_2-1}}_{\Gamma(z_1+z_2)} \right) \cdot \left(\underbrace{\int_0^1 dt\, t^{z_1-1}\, (1-t)^{z_2-1}}_{B(z_1,\, z_2)} \right)$$

$$= \Gamma(z_1)\, \Gamma(z_2) \, .$$

POLIGAMMA

$$\psi^{(n)}(z) = \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z)$$

$$\psi^{(0)}(z) = \psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

"DIGAMMA"

$z \in \mathbb{Z}_{\leq 0} = \{0, -1, -2, \dots\} \to$ POLI SINGOLI DI $\psi^{(0)}(z)$

$\psi^{(n)}(z)$ HA POLI DI ORDINE $n+1$ IN $z \in \mathbb{Z}_{\leq 0}$.

$$* \quad \psi^{(n)}(z+1) = \psi^{(n)}(z) + \frac{(-1)^n \, n!}{z^{n+1}}$$

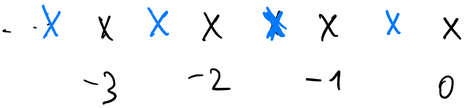
(DIMOSTRARE PER ESERCIZIO)

IDENTITÀ MOLTIPLIC.

K INTERO

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{K}\right)\Gamma\left(z+\frac{2}{K}\right)\cdots\Gamma\left(z+\frac{K-1}{K}\right) = \Gamma(Kz)\,(2\pi)^{\frac{K-1}{2}}\,K^{\frac{1}{2}-Kz}$$

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = \Gamma(2z)\cdot \quad \#$$



$$\log \Gamma(1+z) = -\gamma_E z + \sum_{k=2}^{\infty} \frac{\zeta_k}{k}(-z)^k$$

SERIE DI
TAYLOR IN Z

$\gamma_E \equiv$ COST. DI EULERO - MASCHERONI

$$= \int_0^{\infty} e^{-t} \log t \, dt = 0.577216\ldots\ldots$$

$$\zeta_k = \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \mathrm{Re}\, s > 1$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \cdot \int_0^\infty \frac{t^{s-1}}{(e^t - 1)}\, dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{k=0}^{\infty} e^{-t(k+1)}\, dt$$

$n = k+1$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\frac{1}{e^t - 1} = \frac{e^{-t}}{1 - e^{-t}} = e^{-t} \cdot \sum_{k=0}^{\infty} e^{-t \cdot k}$$

$$t > 0 \Rightarrow |e^t| > 1$$

$$|e^{-t}| < 1$$

$$t = \frac{u}{n}$$

$$\frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \underbrace{\int_0^\infty e^{-t \cdot n}\, t^{s-1}\, dt}$$

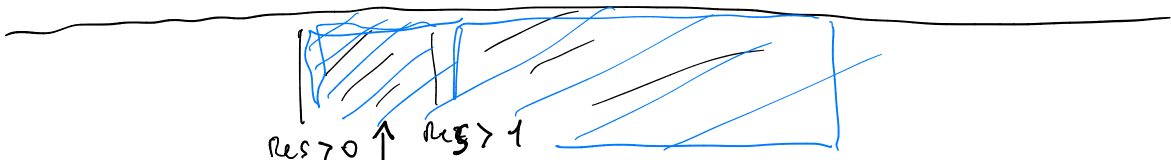
$$\cancel{\frac{1}{\Gamma(s)}} \underbrace{\sum_{n=1}^{\infty} n^{-s+\cancel{1}} \cdot \cancel{n^{-1}}}_{\zeta(s)} \int_0^\infty \underbrace{e^{-u}\, u^{s-1}\, du}_{\cancel{\Gamma(s)}} .$$

$\zeta(s)$ HA L'UNICO POLO PER $s = 1$

$\zeta(s)$ HA INFINITI ZERI:

$$\zeta(-2k) = 0 \quad , \quad k \in \mathbb{N}^+ \qquad \text{"ZERI TRIVIALI"}$$

HA ANCHE NUM. INF. DI ZERI "NON-TRIVIALI"



DOMINIO ESTESO DI CONVERGENZA PER $\eta(s)$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

FUNZ. "ETA DI DIRICHLET"

$$\eta(s) = \zeta(s) - 2 \sum_{n=1}^{\infty} \frac{1}{(2 \cdot n)^s} = \zeta(s) - 2\, 2^{-s} \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$= \zeta(s) \cdot \left(1 - 2^{1-s}\right).$$

* ESERCIZIO: USARE $\zeta(s) = \dfrac{\eta(s)}{(1 - 2^{1-s})}$ PER CALCOLARE $\operatorname{Res} \zeta(s) \Big|_{s=1}$.

$$\eta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t + 1}\, dt$$

COLLEGATO A DISTR. FERMI-DIRAC

RELAZIONE DI RIFLESSIONE

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$$\frac{\zeta(s)}{\zeta(1-s)} = 2^s \pi^{s-1} \cdot \sin\left(\frac{\pi s}{2}\right) \cdot \Gamma(1-s)$$

O ZERI