REGULAR & SINGULAR POINTS

WE WILL CONSIDER IN DETAIL LINEAR ODE'S SUCH AS

$$y''(z) + p(z)\,y'(z) + q(z)\,y(z) = 0$$

WHERE $p(z)$ AND $q(z)$ ARE ANALYTIC FUNCTIONS (APART FOR ISOLATED SINGULARITIES)

IN THIS CASE THE SOLUTION $y(z)$ WILL ALSO BE ANALYTIC, EXCEPT FOR POSSIBLE SINGULARITIES AT THE SAME POINTS WHERE $p(z)$ OR $q(z)$ ARE SINGULAR.

IN THIS DISCUSSION IT IS MORE CONVENIENT TO CONSIDER A COMPLEX VARIABLE $z \in \mathbb{C}$ FOR THE ODE. (WE CAN ALWAYS RESTRICT TO THE REAL CASE IF NEEDED).

LET US CONSIDER THE BEHAVIOUR OF THE GENERAL SOLUTION AROUND A POINT z_0. WE CAN HAVE THREE SITUATIONS.

1) **REGULAR POINT:**

IF $p(z)$ AND $q(z)$ ARE ANALYTIC IN $z = z_0$.

THEN THEY HAVA A CONVERGENT TAYLOR EXPANSION

$$p(z) = \sum_{n=0}^{\infty} p_n (z - z_0)^n , \quad q(z) = \sum_{n=0}^{\infty} q_n (z - z_0)^n .$$

THEN WE CAN ALSO WRITE THE GENERAL SOLUTION AS A TAYLOR SERIES

$$y(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n .$$

THE COEFFICIENTS c_n CAN BE FOUND BY PLUGGING THE EXPANSION INTO THE EQUATION.

THE RADIUS OF CONVERGENCE OF THE TAYLOR SERIES FOR $y(z)$ IS DICTATED BY THE CLOSEST SINGULARITY OF $p(z)$ OR $q(z)$.

REGULAR POINTS ARE POINTS WHERE WE CAN GIVE INITIAL CONDITIONS FOR AN INITIAL VALUE PROBLEM. THEY ARE SIMPLY REFLECTED IN THE SERIES, IN FACT $y(z_0) = c_0$, $y'(z_0) = c_1$.

POINTS WHERE EITHER $p(z)$ OR $q(z)$ HAVE A SINGULARITY ARE CALLED SINGULAR POINTS. IN THIS CASE IN GENERAL WE CANNOT GIVE INITIAL CONDITIONS AT THESE POINTS, BUT WE CAN STILL FIND A BASIS OF SOLUTIONS AROUND THEM.

WE CAN DISTINGUISH TWO TYPES OF SING. POINTS IN OUR CLASSIFICATION.

2) "REGULAR" SINGULAR POINT (OR "FUCHSIAN" SINGULAR POINT)

z_0 IS A FUCHSIAN SINGULARITY WHEN:

$p(z)$ HAS AT MOST A SINGLE POLE AT $z = z_0$

AND $q(z)$ HAS AT MOST A DOUBLE POLE AT $z = z_0$

(WE WILL SEE LATER HOW THIS CONDITION IS MODIFIED WHEN $z_0 = \infty$, HERE z_0 IS FINITE).

IN THIS CASE, A BASIS OF SOLUTIONS CAN BE FOUND IN THE FORM:

$$y_1(z) = (z - z_0)^{\rho_1} \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

$$y_2(z) = (z - z_0)^{\rho_2} \sum_{n=0}^{\infty} d_n (z - z_0)^n$$

(+ possible log - term)

(WE DISCUSS THE LOG SUBTLETY LATER).

SUCH EXPANSIONS ARE CONVERGENT (WITH RADIUS DICTATED BY THE DISTANCE TO THE NEXT SINGULARITY OF $p(z)$ AND $q(z)$).

THEY CAN BE FOUND BY PLUGGING THE EXPANSIONS IN THE ODE, AS WE WILL DO SHORTLY.

THE GENERAL SOLUTION IS A LINEAR COMBINATION OF y_1 AND y_2.

SIDE NOTE

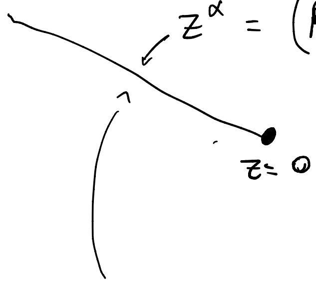
IN GENERAL (FOR $\rho_1, \rho_2 \in \mathbb{Z}$) y_1 AND y_2 HAVE A BRANCH POINT AT $z = z_0$. WE WILL DISCUSS MORE BRANCH POINTS AT THE END OF THE COURSE. THEY ARE POINTS SUCH THAT $f(z)$ CANNOT

BE TAKEN TO BE SINGLE-VALUED IN ANY NEIGHBOURHOOD OF THE POINT $z = z_0$.

* USING RADIAL COORDINATES $z = R e^{i\theta}$, CONVINCE YOURSELF THAT THE FUNCTION $f(z) = z^{\alpha}$ WITH $\alpha \notin \mathbb{Z}$ IS NOT SINGLE-VALUED IN A NEIGHBOURHOOD OF $z = 0$.

WHAT HAPPENS IS THAT GOING AROUND $z = 0$ WE ARE FORCED TO INTRODUCE A DISCONTINUITY ALONG A "BRANCH CUT".

$$z^{\alpha} = \left(R e^{i\theta}\right)^{\alpha} = R^{\alpha} e^{i\theta\alpha}$$



DISCONTINUITY

$$z^{\alpha} = \left(R e^{i(\theta - 2\pi)}\right)^{\alpha} = R^{\alpha} e^{i\theta\alpha} \times e^{-2\pi i \alpha}$$

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* DO THE SAME EXERCISE FOR THE FUNCTION $f(z) = \log(z)$. SHOW THAT IT HAS A BRANCH POINT IN $z = 0$. WHAT IS THE VALUE OF THE DISCONTINUITY?

IN OUR CLASSIFICATION WE FINALLY HAVE THE 3d POSSIBILITY:

3) "IRREGULAR" SINGULAR POINT

z_0 IS AN IRREG. SINGULARITY OF THE ODE IF $p(z)$ HAS A SINGULARITY WORSE THAN A SINGLE POLE AT $z=z_0$, OR $q(z)$ HAS A SINGULARITY WORSE THAN A DOUBLE POLE AT $z=z_0$.

IN THIS CASE, AT LEAST ONE OF THE INDEPENDENT SOLUTIONS OF THE EQUATION WILL HAVE A WORSE BEHAVIOUR AROUND $z=z_0$ THAN THE ONE SEEN SO FAR.

TYPICALLY THIS MEANS IT WILL HAVE AN INFINITE LAURENT SERIES

$$y(z) = \sum_{m=-\infty}^{\infty} c_k (z-z_0)^k$$

, WITH INFINITELY MANY NEGATIVE POWERS. EVEN IF THIS EXPANSION EXISTS, IT IS IN GENERAL

IMPOSSIBLE TO FIND IT RECURSIVELY BY PLUGGING IT INTO THE EQUATION.

SOMETIMES IT IS STILL POSSIBLE TO FIND AN ASYMPTOTIC EXPANSION FOR $y(z)$ AROUND z_0.

WE GAVE THESE DEFINITIONS FOR A 2nd ORDER LINEAR ODE.

THEY GENERALIZE TO THE ORDER n CASE:

$$y^{(n)}(z) + a_{n-1}(z)\, y^{(n-1)}(z) + \dots + a_1(z)\, y'(z) + a_0(z)\, y(z) = 0$$

IN THIS CASE z_0 IS REGULAR IF $a_1(z), \dots, a_{n-1}(z)$ ARE ANALYTIC IN $z = z_0$.

IT IS A FUCHSIAN SINGULARITY IF:

$a_k(z)$ HAS AT MOST A POLE OF ORDER $n - k$ IN $z = z_0$.

e.g. $a_0(z)$ CAN HAVE AN ORDER n POLE.

OTHERWISE, z_0 IS AN IRREGULAR SINGULARITY.

* EXERCISE

CONSIDER $y'(x) + \alpha x^m y(x) = 0$

WHERE α IS A PARAMETER, AND $m \in \mathbb{Z}$.

SOLVE THE ODE EXPLICITLY AND STUDY THE BEHAVIOUR AROUND $x=0$, CHECKING THAT IT HAS THE EXPECTED PROPERTIES WHEN x IS A REGULAR POINT, FUCHSIAN SINGULARITY OR IRREGULAR SINGULARITY.

EXAMPLE : EXPANSION AROUND REGULAR POINT.

AIRY EQUATION: $y''(x) - x\, y(x) = 0$

THIS EQUATION APPEARS IN THE STUDY OF DIFFRACTION OF LIGHT IN THE RAINBOW, AND IN THE W.K.B. METHOD TO SOLVE SCHRÖDINGER EQ. IN QUANTUM MECHANICS.

$x=0$ IS A REGULAR POINT.

WE LOOK FOR SOLUTION $y = \sum_{n=0}^{\infty} c_n x^n$.

(WITH THE INITIAL CONDITION $y(0) = Y_0$, $y'(0) = Y_1$, THEN $c_0 = Y_0$, $c_1 = Y_1$.).

$$y'' = \sum_{n=0}^{\infty} (n)(n-1)\, c_n\, x^{n-2}$$

$$x \cdot y = \sum_{n=0}^{\infty} c_n\, x^{n+1}$$

THE ODE BECOMES (RENAMING INDICES):

$$\underbrace{\sum_{k=0}^{\infty} (k+2)(k+1)\, c_{k+2}}_{y''} x^k - \underbrace{\sum_{k=1}^{\infty} c_{k-1}}_{xy} \cdot x^k = 0$$

WE NEED TO SOLVE SEPARATELY FOR ALL POWERS OF X. THE POWER x^0 IS PRESENT ONLY IN THE FIRST SUM, WITH COEFFICIENT $2 \times C_2$.

THEREFORE WE FIND $\boxed{C_2 = 0}$.

FROM THE POWERS X^K WITH $K \geq 1$, WE FIND:

$$C_{K+2} = \frac{C_{K-1}}{(K+2)(K+1)}, \quad K \geq 1$$

OR EQUIVALENTLY:

$$\boxed{C_{n+3} = \frac{C_n}{(n+3)(n+2)}} \quad (n \geq 0)$$

APPLYING THIS RELATION RECURSIVELY FROM C_{3m} WE FIND:

$$C_{3 \cdot m} = \frac{C_0}{(3m)(3m-1)(3m-3)(3m-4)\cdots\cdots 3 \cdot 2}$$

step of -3

step of -3

$$= \frac{C_0}{(3m)!!!\ (3m-1)!!!}$$

AND IN THE SAME WAY:

$$C_{1+3m} = \frac{C_1}{(3m+1)!!! \quad (3m)!!!}$$

AND $$C_{2+3m} = \frac{\cancel{C_2}}{(3m+2)!!! \, (3m+1)!!!} = 0$$

BECAUSE $C_2 = 0$.

NOTATION

ABOVE $(K)!!!$ MEANS "THE FACTORIAL IN STEPS OF 3".

$$k!!! = K \cdot (K-3)\,(K-6) \ldots$$

stops AT 1, 2, OR 3

SIMILARLY $k!! = K \cdot (K-2) \cdot (K-4) \ldots$

stops at 1 or 2.

NOTICE THAT THIS IS DIFFERENT FROM $(K!)!$

NOW WE HAVE FOUND THE GENERIC COEFFICIENTS.
SO WE HAVE THE FULL SOLUTION!
NOTICE THAT THE ODE DOES NOT HAVE OTHER SINGULARITIES FOR FINITE x. THIS MEANS THAT THE TAYLOR SERIES WE HAVE FOUND WILL CONVERGE EVERYWHERE IN THE COMPLEX PLANE.

LET US SHOW THAT THIS SOLUTION CAN BE IDENTIFIED WITH A COMBINATION OF ${}_pF_q$ FUNCTIONS.

COMMENT: WHEN WE HAVE A POWER SERIES WHERE THE RATIO OF SUCCESSIVE COEFFICIENTS IS A RATIONAL FUNCTION, WE SHOULD ALWAYS TRY TO REWRITE IT IN TERMS OF GENERALISED HYPERGEOMETRIC FUNCTIONS. THEIR DEFINITION IS SO GENERAL THAT THEY CAN ACCOMMODATE SUCH CASES

IN OUR CASE WE HAVE:

$$C_{1+3n} = \frac{C_1}{(1+3n)!!!\ (3n)!!!}, \qquad C_{3m} = \frac{C_0}{(3m)!!!\ (3m-1)!!!}$$

$$C_{2+3m} = 0$$

WE WANT TO REWRITE THIS IN TERMS OF THE $(\alpha)_K$ COEFFICIENTS APPEARING IN THE DEFINITION OF

$${}_pF_q(\alpha_1, \dots \alpha_p ; \beta_1 \dots \beta_q ; x) = \sum_{K=0}^{\infty} \frac{(\alpha_1)_K \cdots (\alpha_p)_K}{K!\,(\beta_1)_K \cdots (\beta_q)_K} x^K$$

WHERE $(\alpha)_K \equiv \underbrace{\alpha \cdot (\alpha+1) \cdot \ \cdots \ (\alpha+K-1)}_{K \text{ terms}}$

IN OUR CASE WE SEE THAT:

$$(3m)!!! = \underbrace{(3m)\,(3m-3) \cdots 6}_{m \text{ terms}} \cdot 3$$

SO THIS EQUALS

$$(3m)!!! = 3^m \cdot m \cdot (m-1) \cdot \ldots \cdot 2 \cdot 1$$

$$= 3^m \cdot m!$$

SIMILARLY:

$$(3m+1)!!! = \overbrace{(3m+1) \cdot (3m-2) \cdot \ldots \cdot (6+4)(3+1)}^{m \text{ terms}}$$

$$= 3^m \cdot \underbrace{\left(m+\tfrac{1}{3}\right) \cdot \left(m-\tfrac{2}{3}\right) \cdot \ldots \cdot \left(3+\tfrac{4}{3}\right)\left(\tfrac{4}{3}\right)}_{m \text{ terms}}$$

$$= 3^m \left(\tfrac{4}{3}\right)_m$$

AND ALSO

$$(3m-1)!!! = \overbrace{(3m-1)(3m-1-3) \cdot \ldots \cdot (2+3) \cdot 2}^{m \text{ terms}}$$

$$= 3^m \cdot \left(3-\tfrac{1}{3}\right) \cdot \left(3-\tfrac{1}{3}-3\right) \cdot \ldots \cdot \left(1+\tfrac{2}{3}\right) \cdot \left(\tfrac{2}{3}\right)$$

$$= 3^m \left(\tfrac{2}{3}\right)_m$$

SO WE FOUND:

$$C_{3m} = \frac{c_0 \; 3^{-2m}}{m! \left(\frac{2}{3}\right)_m}$$

$$C_{3m+1} = \frac{c_1 \; 3^{-2m}}{m! \cdot \left(\frac{4}{3}\right)_m}$$

$$C_{3m+2} = 0$$

THE SOLUTION IS

$$y(x) = \sum_{k=0}^{\infty} c_k \; x^k =$$

(USING EXPRESSIONS ABOVE - CHECK IT!)

$$= c_0 \sum_{m=0}^{\infty} \left(\frac{x^3}{9}\right)^m \cdot \frac{1}{m! \left(\frac{2}{3}\right)_m}$$

$$+ c_1 \cdot x \sum_{m=0}^{\infty} \left(\frac{x^3}{9}\right)^m \frac{1}{m! \left(\frac{4}{3}\right)_m}$$

THESE HAVE THE FORM OF GEN. HYPERGEOM. EXPANSIONS!

$$= c_0 \, {}_0F_1\left(; \tfrac{2}{3} ; \frac{x^3}{9} \right)$$

$$+ c_1 x \cdot {}_0F_1\left(; 4/3 ; \frac{x^3}{9} \right)$$

SO WE FOUND A VERY EXPLICT SOLUTION IN TERMS OF 2 PARAMETERS c_0, c_1, WHICH ENCODE THE INITIAL CONDITION AT $x = 0$.

$\boxed{\text{COMMENT:}}$ WHY IS IT USEFUL TO REWRITE IN TERMS OF ${}_0F_1$?

BECAUSE A LOT IS KNOWN ABOUT THESE FUNCTIONS. WE CAN:

1. • TAKE Mathematica OR Maple (or Wolfram Alpha) AND PLOT THE SOLUTION

2. • GET IMMEDIATE RESULTS FOR COMPLICATED LIMITS, e.g. THE BEHAVIOUR AT

$$x \to \infty$$

IN THIS CASE, THE BEHAVIOUR AT $x \to \infty$ WOULD BE VERY DIFFICULT TO STUDY STARTING FROM THE ODE (WE WILL SEE THAT FOR THE AIRY EQUATION $x = \infty$ IS AN IRREGULAR SINGULARITY).

BUT SINCE THESE FUNCTIONS ARE WELL-KNOWN WE CAN FIND THE BEHAVIOUR EASILY ON THE DIGITAL LIBRARY OF MATHEMATICAL FUNCTIONS OR OTHER RESOURCES (e.g. WOLFRAM FUNCTIONS SITE).

WHAT WE FIND IS THE FOLLOWING.

IT IS USEFUL TO INTRODUCE TWO LINEAR COMBINATIONS OF OUR HYPERGEOMETRIC FUNCTIONS

$$\begin{pmatrix} Ai(x) \\ \\ Bi(x) \end{pmatrix} = \underbrace{\begin{pmatrix} \dfrac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} & -\dfrac{1}{3^{1/2}\,\Gamma\left(\frac{1}{3}\right)} \\ \\ \dfrac{1}{3^{1/6}\,\Gamma\left(\frac{2}{3}\right)} & \dfrac{3^{1/6}}{\Gamma\left(\frac{1}{3}\right)} \end{pmatrix}}_{\text{COEFFICIENTS}} \begin{pmatrix} {}_0F_1\left(;\frac{2}{3};\frac{x^3}{9}\right) \\ \\ x\,{}_0F_1\left(;\frac{4}{3};\frac{x^3}{9}\right) \end{pmatrix}$$

THESE FUNCTIONS HAVE A RICH BEHAVIOUR AS $x \to \infty$.

FOR INSTANCE ON THE REAL AXIS:

$$Ai(x) \sim \frac{e^{-\zeta}}{2\sqrt{\pi}\, x^{1/4}} \qquad \left(\zeta = \frac{2}{3} x^{3/2}\right)$$

$$Bi(x) \sim \frac{e^{+\zeta}}{\sqrt{\pi}\, x^{1/4}} \qquad \text{FOR } x \to +\infty$$

AND $Ai(-x) \sim \frac{1}{\sqrt{\pi}\, x^{1/4}} \cos\left(\zeta - \frac{\pi}{4}\right)$

$$Bi(-x) \sim \frac{1}{\sqrt{\pi}\, x^{1/4}} \left(-\sin\left(\zeta - \frac{\pi}{4}\right)\right)$$

(i.e. THEY OSCILLATE FOR $z \to -\infty$)

SUCH ASYMPTOTIC BEHAVIOUR CAN BE FOUND VERSION OF THE SADDLE-POINT METHOD, STARTING FROM AN INTEGRAL REPRESENTATION OF $Ai(x)$ AND $Bi(x)$.

THE USEFULNESS OF SPECIAL FUNCTIONS

IS THAT WE CAN JUST TAKE THESE RESULTS AND USE THEM IN APPLICATIONS).

COMMENT 2: NOTICE THAT THE SOLUTIONS TO AIRY ODE OSCILLATE AROUND INFINITY. THIS MEANS THAT INFINITY IS AN ESSENTIAL SINGULARITY FOR THE SOLUTION. WE WILL SEE IN FACT THAT $x=\infty$ IS AN IRREGULAR SINGULARITY FOR THIS ODE.

SOLUTION AROUND FUCHSIAN SINGULAR POINTS

AT A REGULAR SINGULARITY:

$$p(z) = \xi^{-1} \sum_{n=0}^{\infty} p_n \cdot \xi^n$$

p_0 = COEFF. OF SINGLE POLE

$$q(z) = \xi^{-2} \sum_{n=0}^{\infty} q_n \xi^n$$

q_0 = COEFF. OF DOUBLE POLE

WITH $\xi = z - z_0$

LOOK FOR SOL. OF THE FORM:

$$y(z) = \xi^{\rho} \cdot \sum_{n=0}^{\infty} c_n \xi^n$$

THEN THE ODE $y'' + p(z)\, y' + q(z)\, y = 0$

IMPLIES:

$$\sum_{k=0}^{\infty} \xi^{\rho+k-2} \cdot \left[(\rho+k)(\rho+k-1) + p_0 (\rho+k) + q_0 \right] c_k$$

$$+ \sum_{k=0}^{\infty} \xi^{\rho+k-2} \sum_{\ell=1}^{k} \left[p_\ell (\rho+k-\ell)\, c_{k-\ell} + q_\ell\, c_{k-\ell} \right] = 0$$

LEADING ORDER $O(\xi^{\rho-2})$ GIVES:

$$\boxed{\rho(\rho-1) + p_0 \rho + q_0 = 0}$$

INDICIAL EQUATION.

IT IS A QUADRATIC EQUATION FIXING TWO SOLUTIONS ρ_1, ρ_2.

BY CONVENTION WE ORDER THEM SUCH THAT

$$\mathrm{Re}(\rho_1) \geq \mathrm{Re}(\rho_2).$$

FOR OUR SOLUTION, WE CHOOSE ONE, ρ_1 OR ρ_2.

THE FOLLOWING ORDERS GIVE: $(k=1,2,\ldots)$

$$c_k \cdot \left[(\rho+k)(\rho+k-1) + p_0(\rho+k) + q_0\right]$$

$$+ \sum_{\ell=1}^{k} \left[p_\ell (\rho+k-\ell)\, c_{k-\ell} + q_\ell\, c_{k-\ell}\right] = 0$$

RECURSIVE EQUATION FIXING c_k IN TERMS OF $c_{k'}$ WITH $k' < k$.

CAN BE USED ORDER BY ORDER TO FIND ALL c's.

IN THIS WAY, IF $\rho_1 - \rho_2 \notin \mathbb{Z}$, WE CAN CONSTRUCT TWO INDEPENDENT SOLUTIONS:

$$y_1(z) = (z-z_0)^{\rho_1} \sum_{k=0}^{\infty} c_k \, (z-z_0)^k$$

WITH COEFF'S FIXED AS EXPLAINED ABOVE, AND (CHOOSING THE OTHER ROOT OF THE INDICIAL EQUATION):

$$y_2(z) = (z-z_0)^{\rho_2} \sum_{k=0}^{\infty} \tilde{c}_k \, (z-z_0)^k \, , \text{ WITH}$$

OTHER COEFFICIENTS $\tilde{c}_k$ CONSTRUCTED IN THE SAME WAY.

GENERAL SOLUTION: LINEAR COMBINATION OF y_1 AND y_2.

<u>SUBTLETY</u>: WHEN $\boxed{\rho_1 = \rho_2 + m}$, $m \in \mathbb{N}^+$, THEN WE CAN CONSTRUCT THE SOLUTION y_1 AS EXPLAINED ABOVE, BUT THE METHOD BREAKS DOWN IF WE TRY TO BUILD y_2 IN THE SAME WAY. THIS IS BECAUSE AT ORDER $O(\xi^{\rho_2+m})$ WE HAVE THE EQUATION:

$$\tilde{c}_m \cdot \left[(\rho_2 + m)(\rho_2 + m - 1) + p_0 (\rho_2 + m) + q_0 \right]$$

$$+ \sum_{k=1}^{m} \left[\text{COMBINATION OF } \tilde{c}_{k-\ell} \right] = 0$$

THIS DOES NOT FIX $\tilde{c}_m$, BECAUSE THE FIRST BRACKET IS NOW $=0$ DUE TO THE FACT THAT $\rho_2 + m = \rho_1$, AND THE EXPRESSION IN THE BRACKET HAS THE FORM OF THE INDICIAL EQUATION.

IN THIS CASE WE SHOULD LOOK FOR A SECOND INDEPENDENT SOLUTION OF THE FORM:

$$\boxed{y_2(z) = \log(z - z_0) \cdot y_1(z) + (z - z_0)^{\rho_2} \cdot \sum_{n=0}^{\infty} d_n \cdot (z - z_0)^n}$$

THE SAME WORKS ALSO WHEN $\rho_1 = \rho_2$.

WE SHOULD LOOK FOR y_1 IN THE STANDARD FORM, AND FOR y_2 IN THE FORM

THE CASE $\rho_1 - \rho_2 \in \mathbb{N}$ WE HAVE DESCRIBED IS SOMETIMES CALLED THE "RESONANT" CASE.

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* OBSERVATION: WE HAVE SEEN IN THE PREVIOUS LECTURE THAT ONE CAN ALSO RECONSTRUCT THE SECOND SOLUTION FROM THE FIRST USING THE FORMULA:

$$y_2(z) = y_1(z) \cdot \int_0^z \frac{e^{-\int^t p(s)\,ds}}{y_1^2(t)} \cdot dt$$

(THIS CAME FROM THE VARIATION OF CONSTANTS METHOD).

ONE CAN SEE FROM THIS EXPRESSION THAT THE $\log(z - z_0)$ TERM IS NEEDED.

* EXERCISE: VERIFY IT.

HINT: USE THE FACT THAT, WHEN

$$\rho_1 = \rho_2 + m, \quad \text{THEN}$$

$$p_0 - 1 = -2\rho_1 - m.$$

POINT AT INFINITY

TO UNDERSTAND THE BEHAVIOUR AT $z = \infty$, ONE CAN CHANGE VARIABLE IN THE ODE:

$z = 1/\eta$, AND STUDY IT AT $\eta \sim 0$.

DOING THIS CHANGE OF VARIABLES,

$$\frac{d^2}{dz^2} y + p(z) \frac{d}{dz} y + q(z)\, y = 0$$

BECOMES:

$$\eta^4 \cdot \frac{d^2 u}{d\eta^2} + \eta^2 \cdot \left(2\eta - p\left(\frac{1}{\eta}\right)\right) \cdot \frac{d}{d\eta} u + q\left(\frac{1}{\eta}\right) \cdot u = 0$$

FOR $u(\eta) \equiv y\left(\frac{1}{\eta}\right)$.

SO $\underline{\eta = 0 \text{ IS}}$: • **REGULAR** IF
$$\begin{cases} p\left(\frac{1}{\eta}\right) \sim 2\eta + O(\eta^2) \\ q\left(\frac{1}{\eta}\right) \underset{\eta \to 0}{\sim} O(\eta^4) \end{cases}$$

- **FUCHSIAN SINGULARITY**

IF
$$\begin{cases} p\left(\frac{1}{\eta}\right) \sim O(\eta) \\ q\left(\frac{1}{\eta}\right) \sim O(\eta^2) \end{cases} \quad , \eta \to 0$$

AND OTHERWISE IT IS AN IRREGULAR SINGULAR POINT.

GOING BACK TO THE ORIGINAL VARIABLE z:

$z \sim \infty$ IS:

- **REGULAR POINT** IF
$$\begin{cases} p(z) \sim \frac{2}{z} + O\left(\frac{1}{z^2}\right) \\ q(z) \sim O\left(\frac{1}{z^4}\right) \end{cases}$$

FOR $z \to \infty$

- **FUCHSIAN SINGULARITY**

IF :
$$\begin{cases} p(z) \sim O\left(\frac{1}{z}\right) \\ q(z) \sim O\left(\frac{1}{z^2}\right) \end{cases}$$

$z \to \infty$

AND OTHERWISE IT IS AN IRREGULAR SINGULARITY.

* EXERCISE VERIFY THAT $z=\infty$ IS AN IRREGULAR SINGULAR POINT FOR THE AIRY EQUATION.