INTRODUCTION TO PERTURBATIVE METHODS

WE HAVE SEEN SO FAR THAT THERE ARE MANY TOOLS TO SOLVE LINEAR ODE'S ANALYTICALLY (EVEN WITH INHOMOGENEOUS TERMS), ESPECIALLY WHEN THE COEFFICIENTS IN THE ODE ARE ANALYTIC FUNCTIONS.

WHAT CAN WE DO IF WE HAVE:

- A NONLINEAR ODE, e.g.

$$y'' + y + \frac{\cos x}{3 + y^2} = 0$$

- A LINEAR ODE WITH NON-ANALYTIC COEFFICIENTS, e.g.

$$y'' + (x^2 + |x|)\, y = 0$$

?

THE PERTURBATION THEORY APPROACH IS TO INTRODUCE A PARAMETER ε SUCH THAT:

- $\varepsilon = 1$ IS THE COMPLICATED ODE WE WANT TO SOLVE
- THE ODE BECOMES SIMPLE WHEN $\varepsilon = 0$ (WE WILL DISCUSS THE CASE WHEN THE ODE AT $\varepsilon = 0$ IS <u>LINEAR</u>)

WE THEN TRY TO BUILD THE SOLUTION AS A SERIES IN ε AND (IDEALLY) TO RESUM IT GOING TO $\varepsilon = 1$.

IN THE EXAMPLES ABOVE, WE CAN STUDY:

$$y'' + y + \varepsilon \frac{\cos x}{3 + y^2} = 0 \longrightarrow \text{LINEAR AT } \varepsilon \to 0.$$

$$y'' + (x^2 + \varepsilon |x|)\, y = 0 \longrightarrow \text{LINEAR AND ANALYTIC AT } \varepsilon \to 0.$$

WHAT HAPPENS IN GENERAL?

SUPPOSE WE INTRODUCE ε SUCH THAT THE ODE IS:

$$L[y] = \varepsilon \cdot r[y]$$

WHERE $L[y]$ IS A LINEAR DIFFERENTIAL OPERATOR, AND $r[y]$ IS A REMAINDER.

(e.g. IN THE FIRST EXAMPLE ABOVE,

$$L[y] = y'' + y, \quad r[y] = \frac{\cos x}{3 + y^2})$$

THEN WE LOOK FOR A SOLUTION OF THE FORM

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$$

THE LEADING ORDER SOLVES

$$L[y_0] = 0$$

, SIMPLE TO SOLVE.

WE ASSUME WE FOUND y_0 SOLVING THE ODE ABOVE, WITH INITIAL OR BOUNDARY CONDITIONS AS APPROPRIATE.

LET US NOW STUDY THE NEXT ORDERS.

WE HAVE (BY LINEARITY):

$$L[y] = \sum_{n=0}^{\infty} \varepsilon^n L[y_n],$$

AND THE REMAINDER $\varepsilon \cdot r[y]$ EXPANDS AS

$$\varepsilon r[y] = \varepsilon r[y_0] + \varepsilon^2 y_1 \cdot r'[y_0] + \frac{\varepsilon^3}{2} \cdot \left(y_1^2 r''[y_0] + 2 y_2 \cdot r'[y_0] \right) + \dots$$

WHERE WE ASSUMED THAT $r[y_0]$ IS REGULAR.

NOTICE THAT THE TERM $\mathcal{O}(\varepsilon^n)$ IN

$\varepsilon \cdot r[y]$ DEPENDS ONLY ON

$y_0, y_1, \dots, y_{n-1}$ BUT NOT y_n.

WE CAN WRITE:

$$\varepsilon \cdot r[y] = \sum_{n=0} \varepsilon^n R(y_0, \dots, y_{n-1})$$

AT ORDER $\mathcal{O}(\varepsilon^n)$, THE ODE BECOMES

$$L[y_n] = R(y_0, \dots, y_{n-1})$$

WHICH IS THE SAME LINEAR ODE, WITH INHOMOGENEOUS TERM DETERMINED BY PREVIOUS PERTURBATIVE ORDERS.

FOR EXAMPLE :

AT ORDER $O(\varepsilon)$ WE HAVE:

$$L[y_1] = \mathcal{R}[y_0]$$

FROM WHICH WE CAN SOLVE FOR y_1, THEN WE GO TO THE NEXT ORDER AND COMPUTE y_2, AND SO ON ....

EXAMPLE

SOLVE THE INITIAL VALUE PROBLEM

$$\boxed{y'' + \varepsilon e^{-x} y = 0}$$

WITH $y(0)=1$
$y'(0)=1$

ORDER $O(\varepsilon^0)$:

$y_0'' = 0$ WITH $y_0(0) = 1$, $y_0'(0) = 1$

SOLUTION: $y_0(x) = 1 + x$.

ORDER $O(\varepsilon)$:

$$\boxed{y_1'' = -e^{-x} y_0 = -e^{-x}(1+x)}$$

WHERE NOW: $y_1(0) = 0$
$y_1'(0) = 0$

SO THAT $y(0) = 1$, $y'(0) = 1$

FOR $y = y_0 + \varepsilon y_1 + O(\varepsilon^2)$

THE SOLUTION IS SIMPLE.

$$y_1'(x) = -\int_0^x e^{-t}(1+t)\,dt \,,$$

WHERE

WE FIXED THE LOWER INTEGRATION LIMIT SO THAT $y_1'(0) = 0$.

INTEGRATING ONCE MORE:

$$y_1(x) = -\int_0^x dt \int_0^t ds\,(1+s)\,e^{-s} \,,$$

WHERE $y_1(0) = 0$ ✓.

AT GENERIC ORDER, WE FIND:

$$L[y_n] = -e^{-x}\, y_{n-1}(x)$$

WITH INITIAL CONDITIONS $y_n(0) = y_n'(0) = 0$ FOR $n \geq 1$.

THE GENERAL SOLUTION IS

$$y_n(x) = \underbrace{\int_0^x dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} \cdots \cdots \int_0^{t_{2n-1}} dt_{2n}}_{2 \cdot n \text{ INTEGRATIONS}} (1+t_{2n})\, e^{-t_{2n}}$$

IN THIS PARTICULAR CASE, IT CAN BE SHOWN THAT THE PERTURBATIVE SERIES CONVERGES. IT GIVES A VERY GOOD

APPROXIMATION TO THE SOLUTION EVEN WITH JUST A FEW TERMS. IT IS MUCH MORE EFFICIENT FOR INSTANCE THAN SOLVING THE ODE AS A SERIES IN x AROUND $x=0$.

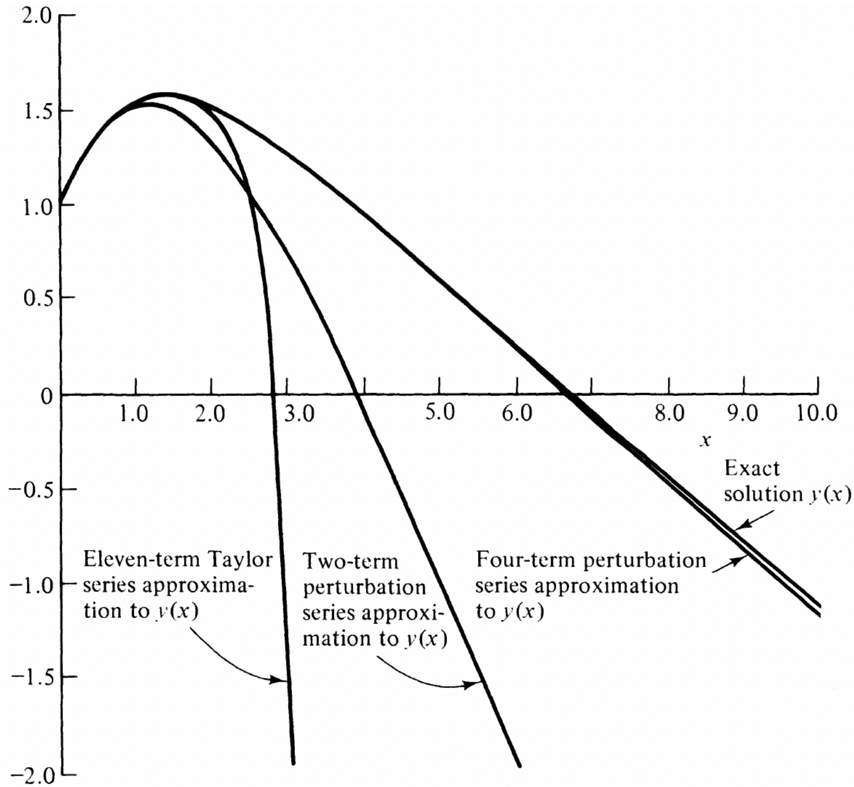


Figure 7.1 A comparison of Taylor series and perturbation series approximations to the solution of the initial-value problem $y'' = -e^{-x}y$ $[y(0) = 1,\ y'(0) = 1]$ in (7.1.15). The exact solution to the problem is plotted. Also plotted are an 11-term Taylor series approximation of the form in (7.1.14) and 2- and 4-term perturbation series approximations of the form in (7.1.3) with $\varepsilon = 1$. The global perturbative approximation is clearly far superior to the local Taylor series.

Figure from Bender and Orszag "Advanced mathematical methods for scientists and engineers", it shows the approximation in the example discussed above

IN GENERAL WE SHOULD BE CAREFUL:

* IT IS NOT GUARANTEED THAT THE PERTURBATIVE SERIES CONVERGES!

- OFTEN IT IS JUST AN ASYMPTOTIC SERIES.

HOWEVER EVEN IN THAT CASE IT CAN BE VERY USEFUL.

OFTEN A FEW TERMS OF AN ASYMPTOTIC SERIES GIVE AN EXCELLENT APPROXIMATION (cf. STIRLING), AS LONG AS WE DO NOT INCLUDE TOO MANY TERMS.

SO FAR WE DISCUSSED EXAMPLES OF "REGULAR" PERTURBATION THEORY.

WE NOW MENTION MORE COMPLICATED CASES WHERE THE PERTURBATION IS "SINGULAR".

THERE ARE CASES WHERE THE LIMIT $\varepsilon \to 0$ IS NOT SMOOTH:

- THE SOLUTION MAY BECOME DISCONTINUOUS, FASTLY OSCILLATING, OR DEVELOP INFINITE GRADIENTS FOR $\varepsilon \to 0^+$

- TYPICALLY THE PROBLEM AT $\varepsilon = 0$ HAS QUALITATIVELY DIFFERENT SOLUTIONS THAN FOR $\varepsilon > 0$, HOWEVER SMALL.

THIS TYPICALLY (BUT NOT EXCLUSIVELY) HAPPENS WHEN ε IS IN FRONT OF THE HIGHEST DERIVATIVE TERM.

IN SUCH CASES WE SAY THE PERTURBATION IS "SINGULAR". IT REQUIRES SPECIAL TECHNIQUES THAT WE WILL NOT COVER IN DEPTH. THE GOAL OF THIS SECTION IS FOR YOU TO BE AWARE OF SINGULAR PERTURBATIONS, SO YOU CAN RECOGNIZE THEM AND FIND INFORMATION IF YOU NEED IN THE FUTURE.

LET US MAKE AN EXAMPLE :

$$\varepsilon y'' + (1+\varepsilon) y' + y = 0 \qquad \begin{aligned} y(0) &= 0 \\ y'(1) &= 1 \end{aligned}$$

THIS ODE I.V.P. HAS A VERY SIMPLE SOLUTION :

$$y(x) = \frac{\varepsilon}{(1-\varepsilon)} \left(e^{-x} - e^{-x/\varepsilon} \right)$$

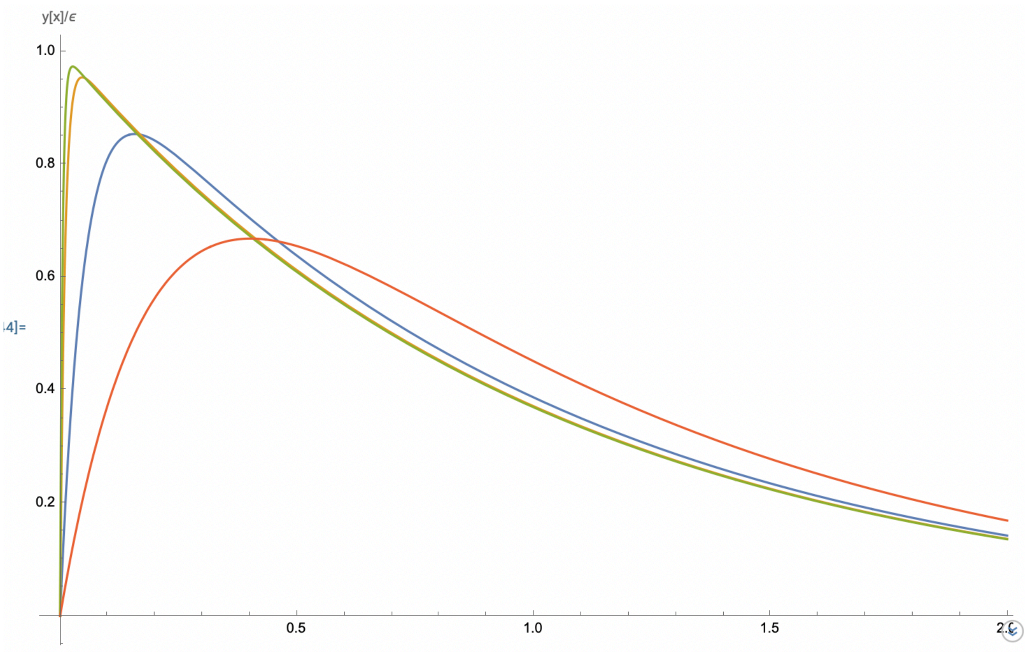
HOWEVER IF WE TRY TO TREAT IT PERTURBATIVELY, WE ENCOUNTER SOME STRANGE FEATURES :

- STRICTLY AT $\varepsilon = 0$, THE IVP BECOMES INCONSISTENT : IN FACT,

  $y' + y = 0$ HAS NO SOLUTION WITH $y(0) = 0$, $y'(0) = 1$!

- A RELATED PHENOMENON IS THAT THE SOLUTION AT $0 < \varepsilon << 1$ DEVELOPS A REGION OF VERY FAST VARIATION (AFTER RESCALING)

↓



Plot of y[x]/epsilon for epsilon = 0.2 (red), 0.05 (blue), 0.01 (orange), 0.005 (green)

SUCH REGION OF FAST VARIATION WHERE $y' \to \infty$ FOR $\varepsilon \to 0^+$ IS CALLED A "BOUNDARY LAYER"

- THERE ARE TECHNIQUES TO TREAT SINGULAR PERTURBATIONS. THE MAIN IDEA IS THAT (IN A CASE LIKE THE ONE ABOVE, SUPPOSING WE DID NOT KNOW THE SOLUTION), WE WOULD NEED TO CONSIDER TWO DIFFERENT REGIONS: THE BOUNDARY LAYER, AND THE OUTER REGION. INSIDE THE BOUNDARY LAYER, IT IS NATURAL TO WORK WITH A RESCALED VARIABLE $z = \frac{x}{\varepsilon}$.

THE TWO REGIONS ADMIT TWO SEPARATE PERTURBATIVE EXPANSIONS, WHICH THEN SHOULD BE GLUED TOGETHER.

MANY TECHNIQUES TO DEAL WITH THIS KIND OF PROBLEMS CAN BE FOUND IN

BENDER & ORSZAG "ADVANCED MATHEMATICAL METHODS FOR SCIENTISTS AND ENGINEERS".

NOTE: OFTEN ε COULD BE A PHYSICAL PARAMETER e.g. $\frac{1}{c^2}$, OR $\hbar^2$, APPEARING IN OUR PROBLEM. THEREFORE WE HAVE NO "CHOICE" ON HOW IT APPEARS IN THE ODE, AND WE MAY HAVE TO CONSIDER SINGULAR PERTURBATIONS FOR PHYSICAL REASONS.

A FAMOUS EXAMPLE OF SINGULAR EXPANSION IS THE WKB APPROXIMATION IN QUANTUM MECHANICS, WHICH CAN BE SEEN AS AN ASYMPTOTIC PERTURBATIVE EXPANSION IN $\hbar \to 0^+$.

$$-\frac{\hbar^2}{2m}\psi'' = (E - V(x))\psi(x)$$

$\hbar$ APPEARS IN A "SINGULAR" WAY IN FRONT OF HIGHEST DERIVATIVE

IN THIS CASE THE REGIONS ANALOGOUS TO BOUNDARY LAYERS ARE THE NEIGHBOURHOODS OF TURNING POINTS OF THE POTENTIAL, i.e. WHERE $V(x) = E$.