EXERCISES ON IRREGULAR SING. PTS.

1 BESSEL EQ.
$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$x=0$: FUCHSIAN PT, INDICES : $\pm\nu$

$x=\infty$: IRREG. SINGULAR POINT

LET US CONSIDER THE ASYMPTOTIC EXPANSION FOR $x \to +\infty$.

TAKING $y(x) = e^{S(x)}$, THE ODE BECOMES

$$x^2 \left(S'' + (S')^2 \right) + x S' + x^2 - \nu^2 = 0$$

SMALLER

SMALLER THAN x^2

WE ASSUME : $|S''| \ll (S')^2$, THEN WE CAN DROP SOME TERMS.

WE THEN SEE THAT $|x S'| \ll x^2 (S')^2$,

OTHERWISE WE WOULD HAVE $S' \ll \frac{1}{x}$, WHICH IMPLIES THAT IT WOULD BE IMPOSSIBLE TO BALANCE THE x^2 TERM.

THEN THE DOMINANT BALANCE IS :

$\downarrow$

$$\boxed{x^2 (S')^2 \sim -x^2}$$

SOLUTIONS : $\quad S(x) \sim \pm i x$

WE CAN NOW VERIFY A POSTERIORI THAT INDEED

$$\underbrace{|S''|}_{O(\frac{1}{x})} \ll \underbrace{(S')^2}_{O(1)}$$

SO OUR ASSUMPTION WAS JUSTIFIED.

IF WE WANT TO GET THE FULL CONTROLLING FACTOR, WE NEED TO GO TO THE NEXT ORDER IN THE EXPANSION :

$$S(x) = \pm i x + C(x), \qquad C(x) \ll x \quad \text{FOR } x \to +\infty.$$

THE ODE BECOMES :

SMALLER   SMALLER

$$x^2 \cdot \left(-\cancel{1} + C'' + (C')^2 \pm 2 i C' \right) + x \cdot \left(\pm i + C' \right) + \cancel{x^2} - \nu^2 = 0$$

$\ll O(x)$ SMALLER

NOW WE SEE THAT AS A CONSEQUENCE OF $C(x) \ll x$, WE HAVE

$$C' \ll O(1) \to (C')^2 \ll C'$$

$$C'' \ll \frac{1}{x} \to x^2 C'' \ll O(x)$$

DROPPING THE SMALLER TERMS :

$$\pm 2i \, x^2 c' \sim \mp i x$$

$$\hookrightarrow \quad c' \sim -\frac{1}{2x} \quad \longrightarrow \quad C(x) \sim \log x^{-\frac{1}{2}}$$

SO WE GET : $S(x) \sim \pm ix - \frac{1}{2} \log x$

IN FACT GOING TO ONE MORE ORDER WE COULD SEE THAT THE REMAINDER IS $\to 0$ FOR $x \to +\infty$

i.e. $S(x) \simeq \pm ix - \frac{1}{2} \log x + \delta(x)$,

$\delta(x) \to 0$ FOR $x \to +\infty$.

THIS IMPLIES THAT

$$y(x) = e^{S(x)} \sim e^{\pm ix} x^{-\frac{1}{2}} .$$

TAKING A LINEAR COMBINATION OF THESE TWO BEHAVIOURS WE CAN INDEED GET

$$\underset{x \to +\infty}{\sim} \begin{cases} \cos(x - \theta) \, x^{-\frac{1}{2}} \\ \text{OR} \\ \sin(x - \theta) \, x^{-\frac{1}{2}} \end{cases}$$

WHERE θ IS ANY SHIFT.

NOTE:
THE "CANONICAL BASIS" $J_\nu(x)$, $Y_\nu(x)$ OF SOLUTIONS OF BESSEL EQ. IS ALSO DEFINED BY THEIR PROPERTIES AT $x \to 0$. (e.g. $J_\nu(x) \underset{x\to 0}{\sim} x^\nu$).

- WHAT ARE THE POSSIBLE BEHAVIOURS FOR $x \to -\infty$?

REPEATING THE ASYMPT. ANALYSIS FOR $x \to -\infty$ WE WOULD FIND THE SAME POSSIBLE BEHAVIOURS:

i.e. $$y(x) \underset{x\to-\infty}{\sim} e^{\pm i x} |x|^{-\frac{1}{2}}$$

NOTE THAT THE $\pm$ SIGN IS NOT CORRELATED WITH HOW THE SOL. BEHAVES AT $+\infty$.

4. EXERCISES ON EXPANDING AT INFINITY.

(WE CONSIDER $x \to +\infty$):

1) $\boxed{y'' = x^{-3} y}$

IN THIS CASE $x = \infty$ IS A **FUCHSIAN POINT**. (REGULAR SINGULARITY!).

IN FACT, $p(x) = 0 \to$ SO IT HAS A BEHAVIOUR NOT WORSE THAN $\sim \frac{\tilde{p}_0}{x}$ FOR $x \to \infty$, WITH $\tilde{p}_0 = 0$.

AND $q(x) = \frac{1}{x^3}$, GOES LIKE $\sim \frac{\tilde{q}_0}{x^2}$ WITH $\tilde{q}_0 = 0$

THE INDICIAL EQ. IS $\rho(\rho+1) = 0$, SO

$\rho = 0$ OR $\rho = -1$.

SOLUTIONS BEHAVE LIKE

$$y_1(x) = \sum_{k=0}^{\infty} a_k \frac{1}{x^k} \quad \text{OR} \quad y_2(x) = A \log x \, y_1(x) + \cancel{x} \sum_{k=0}^{\infty} b_k x^{-k}.$$

2) $\boxed{y'' = x^3 y}$ ← LET US SEE WHAT WOULD CHANGE IN THIS CASE!

IN THIS CASE WE HAVE AN IRREGULAR SING. PT.

SETTING $y = e^{S(x)}$ $\rightarrow$ $S'' + (S')^2 = x^3$

ASSUMING $(S')^2 \gg S''$ FOR $x \to +\infty$,

WE HAVE $(S')^2 \sim x^3$, $x \to +\infty$

$\rightarrow$ $S(x) \sim \pm \frac{2}{5} x^{\frac{5}{2}}$, $x \to +\infty$

CROSS-CHECK: $(S')^2 \sim x^3$

$S'' \sim \pm \frac{3}{2} x^{\frac{1}{2}}$ SO $(S')^2 \gg |S''|$ ✓

NEXT ORDER: $S(x) = \pm \frac{2}{5} x^{5/2} + C(x)$, $C(x) \ll x^{\frac{5}{2}}$ $x \to +\infty$.

$$C'' + (C')^2 \pm \frac{3}{2} x^{\frac{1}{2}} \pm 2 x^{\frac{3}{2}} C' = 0$$

SMALLER

SMALLER

DOMINANT BALANCE: $\frac{3}{2} x^{\frac{1}{2}} \sim -2 x^{\frac{3}{2}} C'$

$$\hookrightarrow \quad C' \sim -\frac{3}{4x} \rightarrow C(x) \sim -\frac{3}{4}\log(x).$$

$$\hookrightarrow \quad y(x) \sim e^{\pm 2/5\, x^{5/2}} \cdot x^{-\frac{3}{4}}$$

(INDEED, THIS IS $<< O(x^{\frac{5}{2}})$).

FOR $x \to +\infty$.

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3) $$x(x-1)\,y'' + y' + \frac{3y}{x-2} = 0$$

- $x = \infty$ IS A FUCHSIAN PT.

INDICIAL EQ:

$$\rho(\rho+1) - \tilde{p}_0\,\rho + \tilde{p}_0 = 0$$

WITH $\tilde{p}_0 = \tilde{q}_0 = 0$.

$$\Longrightarrow \rho = 0 \quad \text{OR} \quad \rho = -1.$$

3) $y'' + x^{-\frac{3}{2}} y' - x^{-2} y = 0.$

$x = \infty$ IS AN IRREGULAR SINGUL. POINT.

→ I STUDY THE EXPANSION FOR $x \to +\infty$.

$$y = e^{S(x)} \rightarrow \boxed{S'' + (S')^2 + x^{-\frac{3}{2}} S' - x^{-2} = 0}$$

IN THIS CASE THE ASSUMPTION $|S''| \ll (S')^2$ IS INCONSISTENT!

MOREOVER, THERE IS NO POSSIBLE $\overset{\text{DOMINANT}}{\vee}$ BALANCE INVOLVING ONLY TWO TERMS THAT IS CONSISTENT.

THE CORRECT DOMINANT BALANCE IS:

$$S'' \sim O((S')^2) \sim O(x^{-2}), \quad x \to +\infty$$

WITH $S' x^{-\frac{3}{2}} \ll S''$

THIS LEADS TO:

$$\boxed{S'' + (S')^2 \sim x^{-2}}$$

TAKING $S' \sim K x^{-1}$

$$S'' \sim -K x^{-2}$$

WE SOLVE FOR K:

$$(-K + K^2) \cancel{x^{-2}} \sim \cancel{x^{-2}}$$

$$\to -K + K^2 = 1 \to \boxed{K_\pm = \frac{1}{2}\left(1 \pm \sqrt{5}\right)}$$

$$\to \boxed{S(x) = K_\pm \log(x) + C(x)}$$

$$C(x) \ll \log(x).$$
$$(x \to +\infty)$$

CONSIDERING ONE MORE ORDER WE CAN PROVE THAT $C(x) \ll 1$ FOR $x \to +\infty$.

PARENTHESIS TO VERIFY THIS

IN FACT, $C(x)$ SATISFIES THE ODE:

$$-K_\pm \cancel{x^{-2}} + C'' + (C')^2 + \cancel{(K_\pm)^2 x^{-2}} + 2 K_\pm C' x^{-1} + x^{-\frac{3}{2}}(K_\pm x^{-1} + C') = \cancel{x^{-2}}$$

$$\rightarrow \quad C'' + \underbrace{(C')^2}_{} + 2K_\pm C' x^{-1} + \underbrace{x^{-\frac{3}{2}} C'}_{} + K_\pm x^{-\frac{5}{2}} = 0$$

SMALLER SMALLER

SINCE $C(x) \ll \log x$ WE SEE THAT SOME TERMS ARE CLEARLY SUBDOMINANT $\rightarrow$ WE CAN DROP THEM.

$$C'' + 2K_\pm C' x^{-1} \sim -K_\pm x^{-\frac{5}{2}}$$

AGAIN A BALANCE OF 3 TERMS.

TAKING $C' \sim \alpha x^{-\frac{3}{2}}$ WE GET

$$C'' \sim -\frac{3}{2}\alpha x^{-\frac{5}{2}}$$

$$-\frac{3}{2}\alpha \cancel{x^{-\frac{5}{2}}} + 2K_\pm \alpha \cancel{x^{-\frac{5}{2}}} \sim -K_\pm \cancel{x^{-\frac{5}{2}}}$$

$$\rightarrow -\frac{3}{2}\alpha + 2K_\pm \alpha = -K_\pm$$

$\longrightarrow$ α IS FIXED.

THIS IS CONSISTENT WITH ALL ASSUMPTIONS. SO WE PROVED $C(x) \sim x^{-\frac{1}{2}} \ll 1$.

END OF PARENTHESIS.

SINCE $C(x) \ll 1$, WE CAN WRITE

$$\boxed{y(x) = e^{S(x)} \sim e^{K_\pm \log(x)} \sim x^{K_\pm} \quad , x \to +\infty.}$$

THUS SOLUTIONS HAVE POWER-LIKE BEHAVIOUR SIMILAR TO THE BEHAVIOR AT A FUCHSIAN POINT!

WHAT MAKES THIS POINT "IRREGULAR" THEN?

THE FACT THAT, IF WE PROCEED BY COMPUTING THE NEXT TERMS, WE FIND AN EXPANSION IN POWERS OF $x^{-\frac{1}{2}}$, RATHER THAN x^{-1}.

$$y_\pm(x) = x^{K_\pm} \cdot \sum_{n=0}^{\infty} a_n \, x^{-\frac{n}{2}} .$$

(AND THIS SERIES IS CONVERGENT!)

IN FACT, THERE IS A SIMPLE EXPLANATION OF THIS FACT: WE CAN CHANGE VARIABLE IN OUR ORIGINAL EQUATION

$$y'' + x^{-\frac{3}{2}} y' - x^{-2} y = 0$$

TAKING $z \equiv x^{\frac{1}{2}}$, AND THIS YIELDS THE ODE

$$z^2 Y''(z) + (2-z) Y'(z) - 4 Y(z) = 0$$

WHERE $z = \infty$ NOW IS A FUCHSIAN POINT IN THE NEW VARIABLE!

IN FACT, YOU CAN VERIFY THAT THE INDICES OF THIS EQ. AROUND $z = \infty$ ARE $\rho_\pm = -2 K_\pm$, WHICH IS CONSISTENT WITH THE EXPANSION FOUND ABOVE WITH A DIFFERENT METHOD.

EXERCISES ABOUT EXPANDING AT A FINITE PT.

7) $$x^2 y'' + (1+3x)y' + y = 0$$

$x=0$ IS AN IRREGULAR SINGULARITY

$\rightarrow$ STUDY EXPANSION FOR $x \rightarrow 0^+$.

$$y = e^{S(x)}$$

$$\hookrightarrow \boxed{x^2\left(S'' + (S')^2\right) + (1+\underbrace{3x}_{\text{drop}})S' + 1 = 0}$$

ASSUME: $S'' \ll (S')^2$

NOTE: FOR $x \rightarrow 0^+$, $3x \ll 1$ $\rightarrow$ $3x$ drop.

WE CAN CHOOSE TWO DOMINANT BALANCES:

$\downarrow$

CASE I :

DOMINANT BALANCE :

$$\boxed{x^2 (S')^2 \sim -S'}$$

WITH $S' \gg 1$, $(S')^2 \gg S''$

SO WE DROP ALSO "1" AND "$x^2 S''$" FROM THE EQ.

WE GET $x^2 S' \sim -1$

$$\rightarrow S(x) \sim \frac{1}{x}$$

CROSS-CHECK : INDEED WITH THIS SOLUTION

$(S')^2 \gg S''$ ✓

AND $S' \gg 1$ ✓

CASE II

AN ALTERNATIVE POSSIBILITY IS :

$(S')^2 \gg S''$

BUT WITHOUT IMPOSING $S' \gg 1$.

THIS MEANS THAT $S' \sim O(1)$ OR $o(1)$,

WHICH IMPLIES $(S')^2 x^2 \ll S'$.

THEN WE MUST DROP THE $x^2 (S')^2$ TERM.

THE DOMINANT BALANCE IS NOW:

$$\boxed{S' \sim -1} \longrightarrow S(x) \sim -x$$

(FOR $x \to 0^+$)

THIS IS CONSISTENT WITH $S'' \ll (S')^2$

AND ALL THE OTHER ASSUMPTIONS.

IN THIS CASE, WE CAN READILY CONCLUDE THAT THERE IS A SOLUTION WITH BEHAVIOUR

$$y_{II}(x) = e^{S(x)} = e^{-x + \text{subleading}}$$

$$\sim 1 - x + o(x)$$

FOR $x \to 0^+$.

WE COULD IN PRINCIPLE DETERMINE ALL TERMS IN A SERIES SOLUTION AROUND 0.

AS MENTIONED IN CLASS, THIS SERIES WOULD BE AN ASYMPTOTIC, BUT NOT CONVERGENT SERIES.

THIS CONCLUDES THE ANALYSIS OF CASE II.

LET US COME BACK TO CASE I, TO CHARACTERIZE THIS OTHER INDEPENDENT SOLUTION.

IN THIS CASE WE HAVE

$$y_I(x) = e^{S(x)}, \quad \text{WITH} \quad S(x) \sim \frac{1}{x}, \quad x \to 0^+.$$

WE CAN WRITE $$S(x) = \frac{1}{x} + C(x),$$

WITH $C(x) << \frac{1}{x}$ FOR $x \to 0^+$.

IF WE WANT TO FIX COMPLETELY THE LEADING BEHAVIOUR OF $y_I(x)$, WE SHOULD ALSO FIX ALL TERMS IN $C(x)$ UNTIL WE FIND TERMS WHICH VANISH AS $x \to 0^+$.

FROM $\boxed{x^2\left(S'' + (S')^2\right) + (1+3x)S' + 1 = 0}$

WE FIND THE ODE FOR $C(x)$:

$$x^2\left(\underbrace{C''}_{\text{drop}} + \frac{2}{x^3} + \frac{1}{x^4} + \underbrace{(C')^2}_{\text{drop}} - 2\frac{C'}{x^2}\right)$$

$$+ (1+3x)\cdot\left(C' - \frac{1}{x^2}\right) + \underbrace{1}_{\text{drop}} = 0$$

SINCE $C' \ll \frac{1}{x^2}$ FOR $x \to 0^+$ WE CAN DROP SOME TERMS:

$$(C')^2 \ll \frac{C'}{x^2}$$

$$C'' \ll \frac{1}{x^3}$$

$$\frac{2}{x} + \cancel{\frac{1}{x^2}} - 2C' + C' + \overbrace{3x\,C'}^{\text{drop}} - \cancel{\frac{1}{x^2}} - \frac{3}{x}$$

$$+ \underbrace{o\left(\frac{1}{x}\right)}_{\text{subleading}} = 0$$

$\left(\ll \frac{1}{x}\right)$

for $x \to 0^+$

WE REMAIN WITH THE DOMINANT BALANCE:

$$-\frac{1}{x} \sim +C' \longrightarrow C(x) \sim -\log x$$

FOR $x \to 0^+$.

SO WE FIND:

$$S(x) \sim \frac{1}{x} - \log x + D(x)$$

WE COULD GO ON AND SHOW THAT

$D(x) \to 0$ FOR $x \to 0^+$.

THEN, THE LEADING BEHAVIOUR OF THIS SOLUTION IS:

$$y_I(x) \sim e^{S(x)} \sim \frac{e^{\frac{1}{x}}}{x} \quad \text{FOR } x \to 0^+$$

IN THIS CASE, IT ACTUALLY TURNS OUT (ACCIDENTALLY) THAT $y(x) = \frac{e^{1/x}}{x}$ IS AN EXACT SOL. OF THE ODE!

5) $y'' = \sqrt{x}\, y$.

$x=0$ IS AN IRREG. SINGULARITY

EXPAND FOR $X \to 0^+$.

$$y = e^{S(x)} \to \left(S'' + (S')^2\right) = \sqrt{x}$$

NOTE: IT IS NOT CONSISTENT TO TAKE $S'' \ll (S')^2$. IN FACT, THIS ASSUMPTION WOULD LEAD TO $S' \sim \pm x^{\frac{1}{4}} \to S \underset{x \to 0^+}{\sim} x^{\frac{5}{4}}$

AND WE WOULD HAVE $S'' \gg (S')^2$!!

LET US TRY THE OPPOSITE: $S'' \gg (S')^2$,

THEN THE DOMINANT BALANCE WOULD BE:

$$\boxed{S'' \sim \sqrt{x}} \to S(x) \underset{x \to 0^+}{\sim} \frac{4}{15} x^{\frac{5}{2}}$$

THIS IS CASE I.

THERE IS ANOTHER POSSIBILITY GIVING A CONSISTENT BALANCE

$$\boxed{S'' \sim -(S')^2} \qquad \text{WITH} \quad (S')^2 \gg \sqrt{x}.$$

THIS HAS A SOLUTION WITH BEHAVIOUR

$$S(x) \sim \alpha \log x$$

FROM THE EQ. ABOVE WE FIX:

$$-\alpha = -\alpha^2 \rightarrow \alpha = 1$$

CROSS-CHECK: INDEED $(S')^2 \sim \frac{1}{x^2} \gg \sqrt{x}$.

THIS IS CASE II.

SO WE HAVE FOUND TWO CONSISTENT BEHAVIOURS:

CASE I: $y_I(x) = e^{S(x)} = e^{\frac{4}{15}x^{5/2}+\dots}$

$$\sim 1 + \frac{4}{15}x^{\frac{5}{2}} + \dots$$

FOR $x \rightarrow 0^+$.

CASE II: $S(x) \sim \log x + C(x)$

$(x \to 0^+)$

WHERE (ANALYZING ONE MORE ORDER) WE COULD SHOW $C(x) \to 0$ FOR $x \to 0^+$.

THEN:

$$y_{II}(x) \sim e^{\log x} \sim x \quad , \text{ FOR } x \to 0^+.$$

IN FACT, CONTINUING WITH THESE EXPANSIONS WE WOULD FIND FURTHER TERMS AS SERIES IN $\sqrt{x}$. THESE SERIES ARE CONVERGENT. IN FACT, ONE CAN VERIFY THAT WITH THE TRANSFORMATION $z = \sqrt{x}$, OUR ODE MAPS TO

$$4 z^3 Y(z) + \frac{Y'(z)}{z} - Y''(z) = 0 \, ,$$ WHERE

$z=0$ IS A FUCHSIAN PT.

THE INDICES ARE $\rho = 0$ AND $\rho = 2$,

WHICH INDEED CORRESPOND TO THE BEHAVIOUR OF

$$y_I \underset{z\to 0}{\sim} z^{\rho=0} \sim \text{const} \quad \text{FOR } x \to 0^+.$$

$$y_{II} \underset{z\to 0}{\sim} z^{\rho=2} \sim x \quad \text{FOR } x \to 0^+.$$

→ THE TWO BEHAVIOURS WE HAVE FOUND WITH THE DOMINANT BALANCE METHOD ✓.

NOTE: IN THIS CASE

$$y'' = \sqrt{x}\, y$$

$x=0$ IS NOT FUCHSIAN BUT IT IS LESS SINGULAR THAN AT A FUCHSIAN PT!

→ IN THIS CASE, THE ASSUMPTION $(S')^2 \gg S''$ DOES NOT WORK → INSTEAD, $(S')^2 \sim S''$ → FUCHSIAN PT IN DIFFERENT VARIABLE!

6

$$x\, y''' - y' = 0$$

IT IS CONVENIENT TO JUST SET

$\psi = y'$ AND STUDY THE ODE:

$$x\, \psi'' - \psi = 0$$

$x=0$ IS A REGULAR SING. POINT

SO WE CAN DETERMINE THE BEHAVIOUR THERE JUST USING THE INDICIAL EQUATION.

$$p_0 = 0 \quad , \quad q_0 = 0$$

$$\rightarrow \rho\,(\rho - 1) = 0 \qquad \text{(INDICIAL EQ.)}$$

$$\rho = 0 \quad \text{OR} \quad \rho = 1.$$

THIS MEANS THERE ARE SOLUTIONS OF THE FORM $\psi_I(x) \sim x \cdot \sum_{n=0}^{\infty} a_n x^n$

AND $\psi_{II}(x) = A\left(\log x\right) y_I(x)$

$$+ \sum_{n=0}^{\infty} b_n x^n$$

WHERE THESE ARE CONVERGENT SERIES $\forall\, x \in \mathbb{C}$.

FROM THIS WE CAN RECONSTRUCT THE POSSIBLE BEHAVIOURS OF $y = \int^{x} \psi(s)\, ds$ AT $x \to 0^+$.

WE HAVE EITHER $y_I(x) \sim x^2$ FOR $x \to 0$

OR: $y_{II}(x) \sim x$ FOR $x \to 0$.

OR WE COULD HAVE A TRIVIAL SOLUTION

$y_{III}(x) = \text{const.}$, WHICH SOLVES THE ODE TRIVIALLY, AND ARISES AS INTEGRATION CONSTANT.

8) 1) $x(x-1)\,y'' + 3y' + \dfrac{y}{x} = 0$

IN THIS CASE $x=0$ IS A REGULAR SINGULARITY.

→ WE CAN JUST STUDY THE INDICIAL EQUATION AT $x=0$.

$$p_0 = -3 \quad \left(= \lim_{x\to 0} x\,p(x) \right)$$

$$q_0 = -1 \quad \left(= \lim_{x\to 0} x^2\,q(x) \right)$$

$$\rho(\rho-1) - 3\cdot\rho - 1 = 0$$

$$\rightarrow \boxed{\rho_\pm = 2 \pm \sqrt{5}}$$

THEN THE SOLUTION CAN HAVE THE BEHAVIOURS

$$y(x) \sim x^{\rho_\pm} \qquad \text{FOR} \quad x \to 0.$$

AND MORE PRECISELY WE CAN FIND CONVERGENT SERIES SOLUTIONS OF THE FORM:

$$y(x) = x^{\rho_\pm} \cdot \sum_{n=0}^{\infty} a_n x^n .$$