

Monetary Economics 2

Notes on: Asset returns, risk and consumption choices: a macroeconomic perspective

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1 Stylized facts and puzzles

Recent empirical research has highlighted a number of regularities, mainly for US data in the post-World-War-II period (but also confirmed over longer samples and extended to other countries by cross-countries studies). The main relevant stylized facts are the following:

1. “high” real average return on stocks (US S&P index: around 8% annually over 1947-2000)
2. “low” real average return on “riskless” bonds \rightarrow riskfree real rate (US T-bills: around 1% annually over 1947-2000)
3. high volatility of real stock returns (annualized standard deviation: around 16%)
4. low volatility of real riskless rate (annualized standard deviation of *ex-post* rate: around 2%)
5. positive and very smooth real consumption growth (for non-durables & services: 2% annual growth rate with 1.1% standard deviation)
6. relatively low correlation between consumption growth and real stock returns (0.23 at quarterly horizon)

This set of facts has several implications for the interpretation of the joint behavior of asset returns and consumption. Mainly:

⇒ facts (1) and (2) imply a high expected excess return on stocks (*high equity premium*)

theory: **CCAPM** (consumption capital asset pricing model) provides an explanation in terms of:

- (i) *covariance* between consumption and stock returns;
- (ii) degree of agents' *risk aversion*

but:

⇒ facts (5) and (6) imply that (i) is low, so a *very high* degree of risk aversion is needed to generate the observed premium:

→ “**equity premium puzzle**” (originally noted by Mehra and Prescott *Journal of Monetary Economics* 1985)

Is the hypothesis of a very high risk aversion consistent with facts (1), (2) and (5) ?

with high risk aversion:

- strong incentive to transfer purchasing power to periods of low expected consumption levels
- given consumption growth (see fact (5)) there should be a tendency for consumers to borrow heavily in capital markets (to transfer consumption from the future to the present), generating an upward pressure on (the general level of) interest rates,

but:

⇒ the relatively low observed interest rate (fact (2)) can be consistent with the positive consumption growth rate only if the consumers' *intertemporal rate of time preference* is very low (even “negative”: agents are very “patient”, favoring future consumption instead of current consumption). Only implausibly low rates of time preference could reconcile consumption growth with low interest rates:

→ “**riskfree rate puzzle**” (observed by Weil *Journal of Monetary Economics* 1989)

2 Standard theory: the Consumption Capital Asset Pricing Model (CCAPM)

2.1 The basic framework

The basic theoretical framework focuses on an infinitely-lived representative agent (consumer/investor) with rational expectations, maximizing an intertemporal utility function defined over consumption flows:

$$\max_{\{C_{t+i}\}} U_t = U(C_t, C_{t+1}, \dots)$$

subject to the dynamic budget constraint:

$$A_{t+i+1} = (1 + r_{t+i}) A_{t+i} + Y_{t+i} - C_{t+i} \quad (i = 0, \dots, \infty)$$

where A_{t+i} is the stock of financial wealth (composed of a single asset yielding a rate of return r_{t+i}) evaluated at the beginning of period $t + i$, Y_{t+i} is the stochastic labour income (exogenously given) and C_{t+i} is consumption; by timing convention, both income and consumption are measured at the end of each period $t + i$.

Several assumptions on the consumer's *preferences* are usually introduced:

- *intertemporal (time) separability*:

$$U(C_t, C_{t+1}, \dots) = v_t(C_t) + v_{t+1}(C_{t+1}) + \dots$$

where $v_{t+i}(C_{t+i}) \equiv$ valuation in t of utility derived from consumption at $t + i \Rightarrow$ “habit formation” and durable goods are ruled out

- future utility *discount* of the form

$$v_{t+i}(C_{t+i}) = \left(\frac{1}{1 + \rho} \right)^i u(C_{t+i})$$

where ρ is the rate of time preference (measuring the degree of the agent's “impatience”) \Rightarrow with this form of “exponential discounting” the possibility of “dynamic inconsistency” of preferences is ruled out

- *expected utility* as objective function (with uncertainty):

$$U_t = E \left(\sum_{i=0}^{\infty} \left(\frac{1}{1 + \rho} \right)^i u(C_{t+i}) \mid I_t \right)$$

\Rightarrow jointly with time-separability, this assumption on U generates an inverse relationship between the *elasticity of intertemporal substitution* and the *degree of risk aversion* (to be seen clearly below in the case of “power utility”).

Under those assumptions the intertemporal maximization problem can be solved. To begin with, let’s consider the simple case of a “safe” financial asset, with known (and constant) rate of return r . In this case the first order condition of the problem (so-called “Euler equation”) is:

$$\text{f.o.c. (for } i = 0) \quad u'(C_t) = \frac{1+r}{1+\rho} E_t u'(C_{t+1})$$

With a *CRRA* utility function (“constant relative risk aversion” or “power utility”) with relative risk aversion (RRA) parameter γ :

$$u(C) = \frac{C^{1-\gamma} - 1}{1-\gamma} \quad \text{with coeff. of RRA } \gamma = -\frac{u''}{u'} C > 0$$

we get:

- (a) if there is *certainty* also on labour income flows Y_{t+i} :

$$C_t^{-\gamma} = \frac{1+r}{1+\rho} C_{t+1}^{-\gamma} \quad \Rightarrow \quad \left(\frac{C_{t+1}}{C_t} \right)^\gamma = \frac{1+r}{1+\rho}$$

in logs, with $c \equiv \log C$:

$$\Delta c_{t+1} = \frac{1}{\gamma} (r - \rho)$$

where $\frac{1}{\gamma}$ measures the *intertemporal elasticity of substitution* (inversely related to the coefficient of RRA)

- (b) with *uncertain* (stochastic) labour income:

$$\frac{1+r}{1+\rho} E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] = 1 \quad \Rightarrow \quad (r - \rho) + \log E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] = 0$$

with distributional assumption:

$$\Delta c_{t+1} \sim N (E_t \Delta c_{t+1}, \sigma_t^2) \quad \Rightarrow \quad -\gamma \Delta c_{t+1} \sim N (-\gamma E_t \Delta c_{t+1}, \gamma^2 \sigma_t^2)$$

using the general property for a (conditionally) lognormally distributed random variable x :

$$\log E_t x = E_t \log x + \frac{1}{2} \text{var}_t (\log x)$$

we get, with $x \equiv \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$

$$(r - \rho) - \gamma E_t \Delta c_{t+1} + \frac{1}{2} \gamma^2 \sigma_t^2 = 0$$

$$\Rightarrow E_t \Delta c_{t+1} = \frac{1}{\gamma} (r - \rho) + \frac{\gamma}{2} \sigma_t^2$$

precautionary savings
effect

A higher degree of uncertainty on labour income (generating a higher consumption variance σ_t^2) induces the agent to reduce current consumption c_t (and increase current savings), with a positive effect on the expected consumption growth rate $E_t \Delta c_{t+1}$: this incentive to save reflects a “precautionary” motive

2.2 The general case: n risky assets

The basic framework can be extended to the general case of n risky financial assets with stochastic returns:

n assets with uncertain returns r^j ($j = 1, \dots, n$)

A_{t+i}^j : stock of asset j held at the beginning of period $t + i$

$A_{t+i} = \sum_{j=1}^n A_{t+i}^j$: stock of financial wealth

r_{t+i+1}^j : return on asset j in period $t + i$ *not known* at the beginning of $t + i \Rightarrow A_{t+i+1}^j = (1 + r_{t+i+1}^j) A_{t+i}^j$

The consumer’s utility maximization problem becomes:

$$\max_{\{C_{t+i}, A_{t+i}^j\}} U_t = E_t \sum_{i=0}^{\infty} \left(\frac{1}{1 + \rho}\right)^i u(C_{t+i})$$

subject to

$$\sum_{j=1}^n A_{t+i+1}^j = \sum_{j=1}^n (1 + r_{t+i+1}^j) A_{t+i}^j + Y_{t+i} - C_{t+i} \quad (i = 0, \dots, \infty)$$

with solution:

$$\begin{aligned} \text{f.o.c. (for } i = 0) \quad u'(C_t) &= \frac{1}{1 + \rho} E_t [(1 + r_{t+1}^j) u'(C_{t+1})] \quad (j = 1, \dots, n) \\ \Rightarrow 1 &= E_t \left[(1 + r_{t+1}^j) \underbrace{\frac{1}{1 + \rho} \frac{u'(C_{t+1})}{u'(C_t)}}_{M_{t+1}} \right] \\ 1 &= E_t [(1 + r_{t+1}^j) M_{t+1}] \end{aligned}$$

where $M_{t+1} = \frac{1}{1 + \rho} \frac{u'(C_{t+1})}{u'(C_t)}$ is the *stochastic discount factor* (marginal rate of intertemporal substitution).

Using the property:

$$E_t [(1 + r_{t+1}^j) M_{t+1}] = E_t (1 + r_{t+1}^j) E_t (M_{t+1}) + \text{cov}_t (r_{t+1}^j, M_{t+1})$$

we can obtain a series of implications on the joint behaviour of financial returns and optimal consumption.

The first implication relates the expected rate of return on each risky asset j to its covariance with the stochastic discount factor:

$$E_t (1 + r_{t+1}^j) = \frac{1}{E_t (M_{t+1})} [1 - \text{cov}_t (r_{t+1}^j, M_{t+1})] \quad (\text{CCAPM 1})$$

Second, if one of the assets is riskless, with certain return r^f , the following property holds:

$$1 + r_{t+1}^f = \frac{1}{E_t (M_{t+1})} \quad (\text{CCAPM 2})$$

Finally, combining CCAPM 1 and CCAPM 2 :

$$\underbrace{E_t r_{t+1}^j - r_{t+1}^f}_{\text{equity premium}} = -(1 + r_{t+1}^f) \text{cov}_t (r_{t+1}^j, M_{t+1}) \quad (\text{CCAPM 3})$$

In the case of “power utility” (CRRA), the first order condition for each asset j becomes:

$$1 = E_t \left[(1 + r_{t+1}^j) \frac{1}{1 + \rho} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right]$$

and in logs:

$$0 = -\rho + \log E_t \left[(1 + r_{t+1}^j) \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right]$$

In order to proceed, now an assumption on the *joint* distribution of the growth rate of consumption and asset returns is necessary. An useful assumption is that the consumption growth rate and the rate of return on each asset j are (conditionally) jointly lognormally distributed. This implies the following property for generic random variables x and y :

$$\log E_t(x_{t+1}y_{t+1}) = E_t(\log(x_{t+1}y_{t+1})) + \frac{1}{2} \frac{\text{var}_t(\log(x_{t+1}y_{t+1}))}{E_t[\log(x_{t+1}y_{t+1}) - E_t(\log(x_{t+1}y_{t+1}))]^2}$$

Applying this property to the expectations term on the right-hand side of the first order condition we obtain

$$\log E_t \left[(1 + r_{t+1}^j) \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] = E_t(r_{t+1}^j - \gamma \Delta c_{t+1}) + \frac{1}{2} \Sigma_j$$

where

$$\Sigma_j = E \left[\left((r_{t+1}^j - \gamma \Delta c_{t+1}) - E_t(r_{t+1}^j - \gamma \Delta c_{t+1}) \right)^2 \right]$$

(the expectation can be made *unconditional* -dropping the subscript t - by assuming conditional homoscedasticity of asset returns and consumption)

$$\Rightarrow E_t r_{t+1}^j = \gamma E_t \Delta c_{t+1} + \rho - \frac{1}{2} \Sigma_j$$

Note:

- (i) the f.o.c. can be expressed as an Euler equation relating the expected consumption growth rate to expected asset returns and the rate of time preference:

$$E_t \Delta c_{t+1} = \frac{1}{\gamma} (E_t r_{t+1}^j - \rho) + \frac{1}{2\gamma} \Sigma_j$$

- (ii) the f.o.c. is a relation between the expected consumption growth rate and the expected returns on *all* assets, given by $\frac{1}{\gamma}$. This delivers a way of empirically testing the theory:

- forecast model for $\Delta c_{t+1} \Rightarrow \Delta c_{t+1} = \boldsymbol{\delta}' \mathbf{x}_t + u_{t+1}$
- estimate the system:

$$\begin{aligned} \Delta c_{t+1} &= \boldsymbol{\delta}' \mathbf{x}_t + u_{t+1} \\ r_{t+1}^j &= \boldsymbol{\pi}'_j \mathbf{x}_t + k_j + v_{t+1}^j \quad (j = 1, \dots, n) \end{aligned}$$

and test restrictions:

$$\boldsymbol{\pi}_j = \gamma \boldsymbol{\delta} \quad \text{for all } j$$

To derive more easily interpretable relations, we calculate the variance term Σ_j :

$$\begin{aligned}
\Sigma_j &= E \left[\left((r_{t+1}^j - \gamma \Delta c_{t+1}) - E_t (r_{t+1}^j - \gamma \Delta c_{t+1}) \right)^2 \right] \\
&= E \left[\left((r_{t+1}^j - E_t r_{t+1}^j) - \gamma (\Delta c_{t+1} - E_t \Delta c_{t+1}) \right)^2 \right] \\
&= \underbrace{E \left[(r_{t+1}^j - E_t r_{t+1}^j)^2 \right]}_{\sigma_j^2} + \underbrace{\gamma^2 E \left[(\Delta c_{t+1} - E_t \Delta c_{t+1})^2 \right]}_{\sigma_c^2} \\
&\quad - 2\gamma \underbrace{E \left[(r_{t+1}^j - E_t r_{t+1}^j) (\Delta c_{t+1} - E_t \Delta c_{t+1}) \right]}_{\sigma_{jc}} \\
&\equiv \sigma_j^2 + \gamma^2 \sigma_c^2 - 2\gamma \sigma_{jc}
\end{aligned}$$

Using this expression for Σ_j the three basic CCAPM relations obtained above become:

$$E_t r_{t+1}^j = \gamma E_t \Delta c_{t+1} + \rho - \frac{\sigma_j^2}{2} - \frac{\gamma^2 \sigma_c^2}{2} + \gamma \sigma_{jc} \quad (\text{CCAPM 1})$$

For the riskfree asset $\sigma_{jc} = \sigma_j^2 = 0$:

$$r_{t+1}^f = \gamma E_t \Delta c_{t+1} + \rho - \frac{\gamma^2 \sigma_c^2}{2} \quad (\text{CCAPM 2})$$

Finally the premia on the risky assets is:

$$\underbrace{(E_t r_{t+1}^j - r_{t+1}^f)}_{\text{adjusted excess return}} + \frac{\sigma_j^2}{2} = \gamma \sigma_{jc} \quad (\text{CCAPM 3})$$

The “premium” equation can be re-interpreted as in Cochrane (2005, p.16):

$$(E_t r_{t+1}^j - r_{t+1}^f) + \frac{\sigma_j^2}{2} = \left(\frac{\sigma_{jc}}{\sigma_c^2} \right) \cdot (\gamma \sigma_c^2) \equiv \beta_{j,\Delta c} \cdot \lambda_{\Delta c}$$

where $\beta_{j,\Delta c}$ has the nature of a “regression coefficient” of the (innovation in the) return on asset j on the (innovation in the) consumption growth rate (similarly to a “beta” coefficient in the simple CAPM model), measuring the “quantity of risk” contained in asset j , whereas $\lambda_{\Delta c}$ is a measure of the “price of risk”, determined by risk aversion and the volatility of consumption. From this perspective, expected returns should be positively related to each asset’s “consumption beta”: for a given “price of risk”, a higher covariance between asset returns and consumption growth increases the asset’s “quantity of risk” and commands a higher expected return to be willingly held. Moreover, a given “quantity of risk” is priced higher the more risk averse investors are (high γ) or the riskier is the environment (higher σ_c^2).

2.3 Quantitative implications

The CCAPM model with a power utility assumption yields several predictions on the relative magnitude of expected returns and risk premia that can be matched with historically observed data. The main findings are the following:

- from CCAPM 3 (with power utility):

$$\underbrace{(E_t r_{t+1}^j - r_{t+1}^f) + \frac{\sigma_j^2}{2}}_{\text{average excess return } \overline{aer}_e} = \gamma \underbrace{\sigma_{jc}}_{\text{cov}(er_e, \Delta c)}$$

given the historically observed values for the average excess return on a risky asset j (or on a stock index) and for the covariance between the excess return and consumption growth, it is possible to compute the value of the risk aversion parameter γ that matches the observed data (called *RRA(1)* in Campbell 2003, Table 4)

\Rightarrow implausibly high values for γ

- from CCAPM 3 rewritten as

$$(E_t r_{t+1}^j - r_{t+1}^f) + \frac{\sigma_j^2}{2} = \gamma \sigma_{jc} = \gamma (\rho_{jc} \sigma_j \sigma_c)$$

where $-1 \leq \rho_{jc} \leq 1$ is the correlation coefficient between (innovations in) consumption growth and returns on asset j (or on a stock index). Assuming $\rho_{jc} = 1$, it is possible to compute the value of γ that matches observed data on average excess returns and consumption growth volatility σ_c (called *RRA(2)* in Campbell 2003, Table 4)

\Rightarrow still implausibly high values for γ

- from CCAPM 2

$$r_{t+1}^f = \gamma E_t \Delta c_{t+1} + \rho - \frac{\gamma^2 \sigma_c^2}{2}$$

the rates of time preference ρ consistent with the values *RRA(1)* and *RRA(2)* for γ can be computed using observed data for the (average) riskfree rate, the average growth rate of consumption and its volatility (the implied values for ρ are called *TPR(1)* and *TPR(2)* by Campbell 2003, Table 5)

\Rightarrow often (implausibly) negative values for ρ

2.4 Insufficient explanations of the equity premium and riskfree rate puzzles

Several potential rationalizations of the “twin” puzzles described above have been put forward in the literature. Though somewhat plausible, all these explanations have problems and cannot entirely account for the detected puzzles. A non-exhaustive list includes:

1. explanations focused on the so-called “*peso problem*”:

a catastrophic event with a small positive probability, not occurred in the observed sample but considered by agents in making consumption/investment choices could explain the very high risk premium required on risky assets (mainly stocks). However, such an event should hit investors in stocks much more than investors in short-term debt instruments to explain a very high premium on equity.

2. explanations based on the so-called “*survival bias*”:

the empirical evidence focuses mainly on the US stock market which is the best performing equity market: this focus could then overstate stock market returns. However the evidence on the equity premium puzzle is found worldwide.

3. explanations based on a “*liquidity effect*”:

greater liquidity of short-term (government) bills relative to longer-term financial assets could explain very low riskfree rates and therefore high equity premia. However, “term premia” are lower than equity premia; even when the short rate is replaced with a long bond rate the evidence of an equity premium puzzle is confirmed.

4. explanations introducing “*individual (idiosyncratic) shocks*” to consumption:

idiosyncratic (and uninsurable) shocks may increase the variability of consumption growth at the *individual* level with respect to the (observable) variability of *aggregate* consumption; if this is

the case, the theoretically relevant σ_c (the consumption volatility faced by the “representative agent”) should be much larger than the value used in quantitative evaluations of the theory. However, this observation does not solve the puzzle since, given $\Delta c_t^i = \Delta c_t + \varepsilon_t^i$ where the individual consumption growth Δc_t^i equals aggregate consumption growth plus an idiosyncratic shock ε_t^i uncorrelated with economy-wide variables (such as stock returns), the covariance between *individual* consumption growth and rates of return is unaffected:

$$\text{cov}(r_t^j, \Delta c_t^i) = \text{cov}(r_t^j, \Delta c_t + \varepsilon_t^i) = \text{cov}(r_t^j, \Delta c_t)$$

3 New research directions

Recent research has explored various ways to extend the basic relations between returns and risk measures in order to account for the observed magnitude of the equity premium and the level of riskfree interest rates.

3.1 More general specification of preferences

The general insight of this class of models is that, in order to account for a high equity premium, additional variables are needed in the utility function that affect marginal utility (and therefore the stochastic discount factor) in a *non-separable* way. Such additional variables may better capture the “state of the economy” (e.g. being a “recession” indicator), which may be relevant to agents in determining their asset allocation choices. Therefore the covariance between equity returns and such additional variables will enter the expression for equilibrium returns and premia on risky assets. As a generic example, given an utility function $u(C, z)$ defined over consumption and an additional “recession” state variable z , the premium on the risky assets becomes:

$$E(r) - r^f = \frac{-C u_{CC}}{u_C} \text{cov}(r, \Delta c) - \frac{z u_{Cz}}{u_C} \text{cov}(r, z)$$

The non-separability of utility in C and z ensures that $u_{Cz} \neq 0$ and the term involving the covariance between the “recession” variable and risky returns contributes to the explanation of the premium over the riskfree asset.

3.1.1 Habit formation

In this vein, one possible extension of preferences which introduces *time non-separability* into the representative agent’s utility function is developed by

Campbell and Cochrane (*JPE* 1999). Their model is focused on the presence of *habit formation* in consumers' behaviour: the level of today's consumption positively affects the marginal utility of tomorrow's consumption. The basic intuition is that people get accustomed over time to a standard of living and a decline in consumption after some time of high consumption (i.e. during a cyclical recession) may hurt more in utility terms.

To capture this form of behaviour an extended utility function is assumed of the form:

$$u(C_t, X_t) = u(C_t - X_t) = \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}$$

where the variable X represents level of "habit" consumption (which depends on aggregate consumption, not affected by the individual agent's choices) and γ is the power parameter (not the risk aversion parameter).

The relation between the current level of consumption and "habit" is captured by the *surplus consumption ratio* $S_t = \frac{C_t - X_t}{C_t}$ (constrained to be positive) so that

$$u_C(C_t, X_t) = u_C(S_t C_t) = (S_t C_t)^{-\gamma}$$

and the risk aversion measure (related to the "curvature" of the utility function) can be obtained as:

$$\frac{-C_t u_{CC}}{u_C} \equiv \eta_t = \frac{\gamma}{S_t}$$

Now risk aversion η_t is not a constant parameter but is time-varying, increasing when the surplus ratio declines: people become more risk averse when consumption falls towards "habit". Then, even a low power coefficient γ is consistent with high (and variable over time) risk aversion. From this model, Campbell and Cochrane (1999) derive several implications

- concerning the *equity premium*: the solution of the representative agent utility maximization problem leads to a first order condition of the form

$$1 = E_t \left[(1 + r_{t+1}^j) \frac{1}{1 + \rho} \left(\frac{S_{t+1}}{S_t} \right)^{-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] \quad j = 1, \dots, n$$

yielding, with convenient distributional assumptions, the following expression for the premium on risky assets

$$\left(E_t r_{t+1}^j - r_{t+1}^f \right) + \frac{\sigma_j^2}{2} = \eta_t \sigma_{jc} \equiv \eta_t (\rho_{jc} \sigma_j \sigma_c)$$

higher risk aversion may explain a high equity premium even in the presence of a low consumption volatility σ_c ;

- concerning the *riskfree rate*: a higher risk aversion does not imply a higher riskfree interest rate (the “riskfree rate puzzle”) due to a strong “precautionary savings” effect:

$$r_{t+1}^f = \gamma E_t \Delta c_{t+1} + \rho - \frac{1}{2} \left(\frac{\gamma}{\bar{S}} \right)^2 \sigma_c^2$$

Uncertainty on consumption levels (captured by σ_c^2) induces consumers to save more: this effect is magnified by the term $\left(\frac{\gamma}{\bar{S}}\right)^2$ (where \bar{S} denotes the average surplus consumption ratio) with a resulting depressing effect on the level of the riskfree rate.

3.2 An alternative multifactor model of risk and returns (Campbell 1996)

Campbell (*JPE* 1996) proposes an intertemporal optimization model which is capable of explaining equilibrium asset returns in terms of various risk factors related not only to the return on the “market” portfolio (as in the basic CAPM model) but also to variables useful to forecast future returns and future labour income. The model is able to account for both the time-series and the cross-section properties of asset returns.

3.2.1 Motivation

Traditional models trying to explain asset returns have used different measures of “risk”. In particular:

- (a) the basic *CAPM* model measures “risk” by means of the covariance of asset returns only with “*market returns*”, usually proxied by the return of a diversified portfolio of common stocks (i.e. a stock market index). Ideally the “market return” should be the return on all the agents’ invested wealth, including non-financial wealth such as human capital.
- (b) the *CCAPM* model uses the covariance of asset returns with *aggregate consumption* as a measure of risk but:
 - aggregate consumption may not be an adequate proxy for consumption of stock market investors. For example, a fraction of the population might be liquidity constrained and not trade in asset markets: the consumption of those agents is irrelevant to the determination of equilibrium asset prices, but is included in aggregate consumption data. Moreover, a fraction of investors

might trade for reasons not related to the optimal intertemporal allocation of consumption and asset portfolio, but for exogenous, psychological reasons: such *noise traders* might influence market prices if rational utility-maximizers investors take into account their presence in making investment choices.

- risk is measured by a covariance with a variable (consumption) which is not exogenous to investors, being determined by their choices; the resulting measure of risk might then be different from the risk perceived by agents.

These weaknesses of traditional models point to the need of an empirically testable model that does not require aggregate consumption data and that derives from theory other factors of risk in addition to the stock index return.

3.2.2 Technique

To develop a model with the above mentioned features, Campbell extends the basic representative agent's intertemporal optimization framework in two main directions, concerning the budget constraint and the utility function:

- (i) a log-linearization of the (intertemporal) budget constraint is used to get closed-form solution for consumption;
- (ii) a more general form of the agent's utility function allows for a coefficient of relative risk aversion unrelated to the elasticity of intertemporal substitution (a more flexible version of the classic "power" utility in which the two measures are inversely related).

The result is an asset pricing formula with no role for consumption but relating asset returns to covariances with the current market return and news about future market returns.

Step 1: log-linearization of budget constraint. The representative agent's one-period budget constraint is written as

$$\underset{\text{total wealth}}{W_{t+1}} = (1 + r_{t+1}^m) \underset{\text{invested wealth}}{(W_t - C_t)}$$

where W includes "human" capital and r^m is the return on the "market portfolio", here including all wealth that is not consumed ("invested wealth" $W - C$). A linear approximation for the budget constraint may be obtained

by first dividing by W_t and taking logs (lower-case variables denote logs of the corresponding upper-case variables):¹

$$\frac{W_{t+1}}{W_t} = (1 + r_{t+1}^m) \left(1 - \frac{C_t}{W_t}\right) \xrightarrow{\text{taking logs}} \Delta w_{t+1} \simeq r_{t+1}^m + \log(1 - e^{c_t - w_t})$$

Now take a first-order Taylor approximation of $\log(1 - e^{c_t - w_t})$ around $\overline{c - w}$ (the mean consumption to wealth ratio, assumed constant in the long-run):

$$\begin{aligned} \log(1 - e^{c_t - w_t}) &\simeq \log(1 - e^{\overline{c - w}}) - \frac{e^{\overline{c - w}}}{1 - e^{\overline{c - w}}} [(c_t - w_t) - (\overline{c - w})] \\ &\Rightarrow \Delta w_{t+1} \simeq r_{t+1}^m + \left(1 - \frac{1}{\rho}\right) (c_t - w_t) + \text{constant} \end{aligned}$$

with $\rho \equiv 1 - e^{\overline{c - w}} = \frac{W - C}{W} < 1$. Using the definition $\Delta w_{t+1} \equiv \Delta c_{t+1} + (c_t - w_t) - (c_{t+1} - w_{t+1})$ we get:

$$c_t - w_t = \rho(c_{t+1} - w_{t+1}) + \rho r_{t+1}^m - \rho \Delta c_{t+1} + \text{constant}$$

Solving forward, with stationary long-run $c - w$, i.e. $\lim_{j \rightarrow \infty} \rho^j (c_{t+j} - w_{t+j}) = 0$, and taking expectations in t we obtain

$$c_t - w_t = E_t \left[\sum_{j=1}^{\infty} \rho^j (r_{t+j}^m - \Delta c_{t+j}) \right] + \text{constant}$$

In this form, the intertemporal budget constraint says that a high consumption to wealth ratio at t is due either to high expected future returns on invested wealth or to expected future low consumption growth. Using this form of the budget constraint the following expression for unexpected changes in consumption is derived:²

$$\begin{aligned} c_{t+1} - E_t c_{t+1} &= (r_{t+1}^m - E_t r_{t+1}^m) + (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{t+1+j}^m - \\ &\quad - (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j \Delta c_{t+1+j} \end{aligned} \tag{1}$$

¹Note that

$$1 - \frac{C_t}{W_t} = 1 - e^{\log(\frac{C_t}{W_t})} \equiv 1 - e^{c_t - w_t}$$

²Note that

$$w_{t+1} - E_t w_{t+1} = r_{t+1}^m - E_t r_{t+1}^m$$

A positive surprise in consumption today corresponds to a higher than expected return on wealth today (first term on the right-hand side) or to news on higher future returns (second term) or to downward revisions in expected future consumption growth (third term). Note that both the above expressions are derived from approximations of the budget constraint and do not impose any behavioural assumptions.

Step 2: generalization of preferences. The representative agent's utility function is of the Epstein-Zin (*JPE*, 1991) form:

$$U_t = \left[(1 - \beta) C_t^{\frac{1-\gamma}{\theta}} + \beta (E_t U_{t+1}^{1-\gamma})^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}}$$

where utility at time t is defined in a recursive way, depending on current consumption, C_t , and the expected value of next period's utility $E_t U_{t+1}$. The parameters involved in this definition are

$$\left. \begin{array}{l} \gamma = \text{RRA coefficient} \\ \sigma = \text{elast. of intert. substitution} \end{array} \right\} \theta = \frac{1 - \gamma}{1 - \frac{1}{\sigma}}$$

so that this formulation of utility allows a choice of the risk aversion parameter independently of the magnitude of the elasticity of intertemporal substitution. As a special case, if $\gamma = \frac{1}{\sigma}$ then $\theta = 1$ and the utility function reduces to the classic "power utility" form. Moreover, β is a preference discount factor (with notation used in previous sections: $\beta \equiv \frac{1}{1+\rho}$).

The agent's utility maximization problem yields the following solutions:

1. for the *market portfolio*:

$$\text{f.o.c. } 1 = E_t \left[\left\{ \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} (1 + r_{t+1}^m) \right\}^{\theta} \right]$$

that, assuming conditional joint lognormal (and homoscedastic) distribution for r^m and C_{t+1}/C_t (usual procedure), reduces to:

$$E_t \Delta c_{t+1} = \sigma E_t r_{t+1}^m - \underbrace{\sigma \log \beta + \frac{1}{2} \frac{\theta}{\sigma} \text{var}_t (\Delta c_{t+1} - \sigma r_{t+1}^m)}_{\text{assumed constant } \mu_m \text{ (under homoscedasticity)}} \quad (*)$$

This is the time-series Euler equation relating expected consumption growth to the expected return on the market portfolio (this relationship is measured by the elasticity of intertemporal substitution σ).

2. from the (more complicated) first order conditions for each *individual risky asset* i , the following (adjusted) risk premia are derived:

$$\left(E_t r_{t+1}^i - r_{t+1}^f\right) + \frac{V_{ii}}{2} = \theta \frac{V_{ic}}{\sigma} + (1 - \theta) V_{im} \quad (2)$$

where

$$\begin{aligned} V_{ii} &\equiv \text{var} (r_{t+1}^i - E_t r_{t+1}^i) \\ V_{ic} &\equiv \text{cov} (r_{t+1}^i - E_t r_{t+1}^i, c_{t+1} - E_t c_{t+1}) \\ V_{im} &\equiv \text{cov} (r_{t+1}^i - E_t r_{t+1}^i, r_{t+1}^m - E_t r_{t+1}^m) \end{aligned}$$

The expected adjusted excess return on asset i is given by a weighted average of two covariances: the covariance of the asset i 's return with consumption divided by σ (with weight θ) and the covariance of the asset i 's return with the market return (with weight $1 - \theta$). Note that:

- (a) with power utility ($\gamma = \frac{1}{\sigma}$ and $\theta = 1$) adjusted risk premia reduce to γV_{ic} as in the standard *CCAPM*;
- (b) with $\gamma = 1$ and then $\theta = 0$, a loglinear version of the static *CAPM* model is obtained, where risk premia are determined by the asset return's covariance with the market return.

Step 3: eliminating consumption. First, from (*) we can construct the revisions in expectations of future consumption growth:

$$\begin{aligned} (E_{t+1} - E_t) \Delta c_{t+j} &= \sigma (E_{t+1} - E_t) r_{t+j}^m \\ \Rightarrow (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j \Delta c_{t+1+j} &= \sigma (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{t+1+j}^m \end{aligned}$$

Then, substituting the last equation into the expression for the innovation in consumption $c_{t+1} - E_t c_{t+1}$ derived from the linearized budget constraint (1) we obtain:

$$c_{t+1} - E_t c_{t+1} = (r_{t+1}^m - E_t r_{t+1}^m) + (1 - \sigma) (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{t+1+j}^m$$

Now innovations in consumption are expressed only in terms of current unexpected market returns (the first right-hand side term, with a one-to-one effect on consumption) and revisions in expectations of future market returns

(the second term). The effect of an increase in expected future returns on consumption depends on the relative strength of the substitution and income effects. If the intertemporal elasticity of substitution σ is less than 1 the agent is unwilling to substitute over time and the income effect of higher expected returns dominates: current consumption increases. If $\sigma > 1$ the substitution effect dominates and current consumption decreases, whereas consumption is unaffected in the special case of $\sigma = 1$ in which the income and substitution effects exactly offset each other.

This equation for the innovation in consumption implies that the covariance of the unexpected return on asset i with $c_{t+1} - E_t c_{t+1}$ (V_{ic}) may be expressed as:

$$V_{ic} \equiv \text{cov} (r_{t+1}^i - E_t r_{t+1}^i, c_{t+1} - E_t c_{t+1}) = V_{im} + (1 - \sigma) V_{ih}$$

where $V_{ih} \equiv \text{cov} \left(r_{t+1}^i - E_t r_{t+1}^i, (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{t+1+j}^m \right)$ measures the covariance between the return on asset i and “news” about future market returns, capturing an intertemporal “hedging” component of asset demand. Finally, using this expression for V_{ic} , the following *cross-sectional asset pricing* formula is derived:

$$\left(E_t r_{t+1}^i - r_{t+1}^f \right) + \frac{V_{ii}}{2} = \gamma V_{im} + (\gamma - 1) V_{ih} \quad (3)$$

Now assets are priced (and therefore expected excess returns are determined) with no direct reference to consumption growth. The risk premium on asset i is a weighted average of two covariances: the covariance of the asset’s return with the return on the market portfolio (including all invested wealth) with weight γ , and the covariance of the asset’s return with news about future market returns with weight $\gamma - 1$. The effect of the latter covariance on risk premia depends on the magnitude of the risk aversion parameter: a positive covariance of an asset return with good news on future market returns reduces the asset ability to “hedge” the investor’s portfolio against adverse changes in investment opportunities; when risk aversion is sufficiently high ($\gamma > 1$) this effect increases the riskiness of the asset and must be compensated by a higher return. Note that the simple logarithmic version of the static *CAPM*, in which investors disregard intertemporal hedging considerations, holds either when $\gamma = 1$ or when $V_{ih} = 0$ for all assets (i.e. in the absence of intertemporal risk), or when V_{ih} is proportional to V_{im} (a case where intertemporal risk is perfectly correlated with market risk across assets), this last case being the empirically relevant one.

Step 4: introducing human capital. To make the model testable, the rate of return on all invested wealth (including the “human” component),

r^m , must be specified in terms of observable variables. With approximation, the market return can be expressed as an average of the returns on financial assets and human capital as:

$$r_{t+1}^m = (1 - v) r_{t+1}^a + v r_{t+1}^y + \text{constant}$$

with r^a = return on financial assets, r^y = return on human wealth (unobservable but proxied by labour income), and $v \simeq 2/3$ is the average ratio of human to total wealth (proxied by the labour share in total income). Using log-linear approximation techniques the following expression for the innovation in the return on human capital is derived:

$$r_{t+1}^y - E_t r_{t+1}^y = (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta y_{t+1+j} - (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{t+1+j}^a$$

A positive innovation in r^y might be due to increases in current and expected future labour incomes Δy (viewed as “dividends” on human capital) or to decreases in expected future financial returns (used to discount future income streams to the present). Using this expression and the above approximation for the overall market return r^m the innovation in consumption (1) can be expressed as:

$$\begin{aligned} c_{t+1} - E_t c_{t+1} &= (1 - v) (r_{t+1}^a - E_t r_{t+1}^a) + v (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta y_{t+1+j} \\ &\quad + (1 - \sigma - v) (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{t+1+j}^a \end{aligned} \quad (4)$$

Now consumption is affected by unexpected changes in the current interest rate r^a (first term), by news on changes in current and future labour income streams (second term) and by news on future interest rates (the third term). The latter effect works through two channels: first, increases in expected future interest rates reduce the present value of human wealth and tend to decrease consumption (this effect works through the coefficient $-v$); second, there is an intertemporal substitution effect which depends on whether σ is less or greater than 1 (see the comment to equation (1)).

Finally, the following final form for the risk premium is obtained:

$$\left(E_t r_{t+1}^i - r_{t+1}^f \right) + \frac{V_{ii}}{2} = \gamma (1 - v) V_{ia} + \gamma v V_{iy} + [\gamma(1 - v) - 1] V_{ih} \quad (5)$$

where

$$\begin{aligned} V_{ia} &\equiv \text{cov}(r_{t+1}^i - E_t r_{t+1}^i, r_{t+1}^a - E_t r_{t+1}^a) \\ V_{iy} &\equiv \text{cov} \left(r_{t+1}^i - E_t r_{t+1}^i, (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta y_{t+1+j} \right) \end{aligned}$$

This expression for the premium has the form of a “multifactor” pricing formula (as in the “arbitrage pricing theory” *APT* models), with intertemporal optimization theory suggesting the nature of the relevant “factors” and imposing a set of restrictions on their “prices” (i.e. the weights with which they enter the risk premium expression). Three factors are relevant here: the financial (stock) market return (through V_{ia}), labour income growth (through V_{iy}) and future changes in investment opportunities (through V_{ih}). Considering human wealth in the determination of equilibrium asset prices can be important since, given a plausible value for $v = 2/3$, the weight on the labour income factor is higher than the weight on the stock market factor. Note also that in order to derive the simplified case of a (logarithmic) static *CAPM* model, now it is necessary to impose both $\gamma = 1$ and $v = 0$, therefore assuming no human capital in the overall investors’ portfolio.

3.2.3 Derivation of a *VAR* factor model

The main implication derived from the theoretical model suggests that relevant “factors” must be selected according to their ability to forecast future stock returns and future labor incomes and their prices must reflect the relative forecasting power of the factors.

To this aim, a *VAR* model can be set up and used to produce forecasts of stock market returns and labour income. Define a K -variable vector \mathbf{z}_t as

$$\mathbf{z}_t = \begin{pmatrix} r_t^a \\ \Delta y_t \\ z_{3t} \\ z_{4t} \\ \dots \\ z_{Kt} \end{pmatrix}$$

where z_3, \dots, z_K are variables potentially relevant to forecast r^a and Δy and in the agents’ information set at t . A *VAR*(1) model of the form

$$\mathbf{z}_{t+1} = \mathbf{A} \mathbf{z}_t + \underbrace{\boldsymbol{\varepsilon}_{t+1}}_{\text{innovations}}$$

can be used to produce forecasts as

$$E_t \mathbf{z}_{t+1+j} = \mathbf{A}^{j+1} \mathbf{z}_t$$

and revisions in expectations between time t and $t + 1$ as, in the case of \mathbf{z}_{t+2} :

$$\left. \begin{aligned} E_{t+1} \mathbf{z}_{t+2} &= E_{t+1} [\mathbf{A} \mathbf{z}_{t+1} + \boldsymbol{\varepsilon}_{t+2}] = \mathbf{A}^2 \mathbf{z}_t + \mathbf{A} \boldsymbol{\varepsilon}_{t+1} \\ E_t \mathbf{z}_{t+2} &= E_t [\mathbf{A}^2 \mathbf{z}_t + \mathbf{A} \boldsymbol{\varepsilon}_{t+1} + \boldsymbol{\varepsilon}_{t+2}] = \mathbf{A}^2 \mathbf{z}_t \end{aligned} \right\} \Rightarrow (E_{t+1} - E_t) \mathbf{z}_{t+2} = \mathbf{A} \boldsymbol{\varepsilon}_{t+1}$$

and in general

$$(E_{t+1} - E_t) \mathbf{z}_{t+1+j} = \mathbf{A}^j \boldsymbol{\varepsilon}_{t+1}$$

Using the vectors $\mathbf{e}_1 = (1 \ 0 \ \dots \ 0)'$ and $\mathbf{e}_2 = (0 \ 1 \ 0 \ \dots \ 0)'$ to extract the first and second element of K -dimensional vectors, we can construct the revisions in expectations (“news”) of future r^a and current and future Δy as:

$$\begin{aligned} (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{t+1+j}^a &= \mathbf{e}'_1 \sum_{j=1}^{\infty} \rho^j \mathbf{A}^j \boldsymbol{\varepsilon}_{t+1} \\ &= \underbrace{\mathbf{e}'_1 \rho \mathbf{A} [\mathbf{I} - \rho \mathbf{A}]^{-1}}_{\boldsymbol{\lambda}'_h} \boldsymbol{\varepsilon}_{t+1} \\ &= \boldsymbol{\lambda}'_h \boldsymbol{\varepsilon}_{t+1} \end{aligned}$$

and

$$\begin{aligned} (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta y_{t+1+j} &= \mathbf{e}'_2 \sum_{j=0}^{\infty} \rho^j \mathbf{A}^j \boldsymbol{\varepsilon}_{t+1} \\ &= \underbrace{\mathbf{e}'_2 [\mathbf{I} - \rho \mathbf{A}]^{-1}}_{\boldsymbol{\lambda}'_y} \boldsymbol{\varepsilon}_{t+1} \\ &= \boldsymbol{\lambda}'_y \boldsymbol{\varepsilon}_{t+1} \end{aligned}$$

with elements of vectors $\boldsymbol{\lambda}_h$ and $\boldsymbol{\lambda}_y$ (being non-linear functions of the coefficients of the VAR matrix \mathbf{A}) measuring the importance of each variable in \mathbf{z} in forecasting future stock market returns and current and future labour incomes respectively.

Now, using the definition

$$V_{ik} \equiv \text{cov} \left(r_{t+1}^i - E_t r_{t+1}^i, \begin{matrix} \varepsilon_{t+1}^k \\ \text{\small } k\text{th element of} \\ \text{\small } \text{vector } \boldsymbol{\varepsilon} \end{matrix} \right)$$

so that $V_{i1} = V_{ia}$, asset pricing equations may be rewritten in a K -factor (“ APT ”) form:

$$\left(E_t r_{t+1}^i - r_{t+1}^f \right) + \frac{V_{ii}}{2} = \gamma (1 - v) V_{i1} + \sum_{k=1}^K \left\{ \gamma v \lambda_y^k + [\gamma(1 - v) - 1] \lambda_h^k \right\} V_{ik}$$

(where λ_y^k and λ_h^k are the k -th elements of vectors $\boldsymbol{\lambda}_y$ and $\boldsymbol{\lambda}_h$), with basic structure

$$\left(E_t r_{t+1}^i - r_{t+1}^f \right) + \frac{V_{ii}}{2} = \sum_{k=1}^K \begin{matrix} p_k \\ \text{\small “prices” of} \\ \text{\small } \text{risk factors} \end{matrix} V_{ik}$$

Each asset i 's excess return is expressed as a linear combination of the covariances of the return on asset i with the innovations in a set of variables useful to forecast stock market returns and labour income and interpreted here as “risk factors”. The coefficients in the linear combinations (p_k) are the “prices” of such factors and are functions of the *VAR* coefficients, the parameters of risk aversion γ , and the share of labour income v . A factor has a high risk price if it is a good forecaster of stock returns or labour income. On the whole, the intertemporal optimization model yields a factor asset pricing model with risk factor prices related to the ability of factors to forecast future returns on both financial wealth and human capital.