VAR modelling for monetary policy analysis: Appendix

Innovation accounting: a bivariate example

Consider a stationary bivariate VAR system in $VMA(\infty)$ representation (omitting constant terms for simplicity):

$$\mathbf{y}_t = \mathbf{\Phi}_0\,\mathbf{v}_t + \mathbf{\Phi}_1\,\mathbf{v}_{t-1} + \mathbf{\Phi}_2\,\mathbf{v}_{t-2} + ... + \mathbf{\Phi}_s\,\mathbf{v}_{t-s} + ...$$

written in full form as:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{0,11} & \phi_{0,12} \\ \phi_{0,21} & \phi_{0,22} \end{pmatrix} \begin{pmatrix} v_{1,t} \\ v_{2,t} \end{pmatrix} + \begin{pmatrix} \phi_{1,11} & \phi_{1,12} \\ \phi_{1,21} & \phi_{1,22} \end{pmatrix} \begin{pmatrix} v_{1,t-1} \\ v_{2,t-1} \end{pmatrix} + $$
$$+ \begin{pmatrix} \phi_{2,11} & \phi_{2,12} \\ \phi_{2,21} & \phi_{2,22} \end{pmatrix} \begin{pmatrix} v_{1,t-2} \\ v_{2,t-2} \end{pmatrix} + ... + \begin{pmatrix} \phi_{s,11} & \phi_{s,12} \\ \phi_{s,21} & \phi_{s,22} \end{pmatrix} \begin{pmatrix} v_{1,t-s} \\ v_{2,t-s} \end{pmatrix} + ...$$

Impulse response functions

The elements of the matrices $\mathbf{\Phi}_i$ trace out the effects over time (*impulse response functions*) of each structural disturbance in $\mathbf{v}$ keeping all the other disturbances at zero, under the set of identifying assumptions used ($\mathbf{\Phi}_0 = \mathbf{A}^{-1}$). Each matrix $\mathbf{\Phi}_s$ captures the effect of the structural shocks at time t on the endogenous variables at time $t+s$; the typical element

$$\phi_{s,ij} = \frac{\partial\, y_{i,t+s}}{\partial\, v_{j,t}}$$

captures the response of the ith element of $\mathbf{y}_{t+s}$ to an "impulse" due to the jth element of $\mathbf{v}_t$. Noting that $\frac{\partial\, y_{i,t+s}}{\partial\, v_{j,t}} = \frac{\partial\, y_{i,t}}{\partial\, v_{j,t-s}}$, from the VMA representation above we get:

- IRF of y_1 to v_1: $\phi_{0,11}, \phi_{1,11}, \phi_{2,11}, ...$ (elements in position $1,1$ in $\mathbf{\Phi}_0, \mathbf{\Phi}_1, \mathbf{\Phi}_2, ...$)
- IRF of y_1 to v_2: $\phi_{0,12}, \phi_{1,12}, \phi_{2,12}, ...$ (elements in position $1,2$ in $\mathbf{\Phi}_0, \mathbf{\Phi}_1, \mathbf{\Phi}_2, ...$)
- IRF of y_2 to v_1: $\phi_{0,21}, \phi_{1,21}, \phi_{2,21}, ...$ (elements in position $2,1$ in $\mathbf{\Phi}_0, \mathbf{\Phi}_1, \mathbf{\Phi}_2, ...$)
- IRF of y_2 to v_2: $\phi_{0,22}, \phi_{1,22}, \phi_{2,22}, ...$ (elements in position $2,2$ in $\mathbf{\Phi}_0, \mathbf{\Phi}_1, \mathbf{\Phi}_2, ...$)

Forecast error variance decomposition

From the $VMA(\infty)$ representation of the VAR it is possible to obtain the forecast of future $\mathbf{y}'$s over an h-period horizon on the basis of information in current (time t) and past values of the variables in the system $\mathbf{y}_t, \mathbf{y}_{t-1}, ...$ The associated forecast error is:

$$\mathbf{y}_{t+h} - E\left(\mathbf{y}_{t+h}\,|\mathbf{y}_t, \mathbf{y}_{t-1}, ...\right) = \mathbf{\Phi}_0\,\mathbf{v}_{t+h} + \mathbf{\Phi}_1\,\mathbf{v}_{t+h-1} + ... + \mathbf{\Phi}_{h-1}\,\mathbf{v}_{t+1}$$

written in full form as

$$\begin{pmatrix} y_{1,t+h} - E\left(y_{1,t+h}\,|\mathbf{y}_t, \mathbf{y}_{t-1}, ...\right) \\ y_{2,t+h} - E\left(y_{2,t+h}\,|\mathbf{y}_t, \mathbf{y}_{t-1}, ...\right) \end{pmatrix} = \begin{pmatrix} \phi_{0,11} & \phi_{0,12} \\ \phi_{0,21} & \phi_{0,22} \end{pmatrix} \begin{pmatrix} v_{1,t+h} \\ v_{2,t+h} \end{pmatrix} + $$
$$+ \begin{pmatrix} \phi_{1,11} & \phi_{1,12} \\ \phi_{1,21} & \phi_{1,22} \end{pmatrix} \begin{pmatrix} v_{1,t+h-1} \\ v_{2,t+h-1} \end{pmatrix} + ... + \begin{pmatrix} \phi_{h-1,11} & \phi_{h-1,12} \\ \phi_{h-1,21} & \phi_{h-1,22} \end{pmatrix} \begin{pmatrix} v_{1,t+1} \\ v_{2,t+1} \end{pmatrix}$$

The *forecast error variance* is given by the following symmetric matrix

$$var\ \left(\mathbf{y}_{t+h} - E\left(\mathbf{y}_{t+h}\left|\mathbf{y}_t, \mathbf{y}_{t-1}, ...\right.\right)\right) = \mathbf{\Phi}_0\,\mathbf{D}\,\mathbf{\Phi}_0' + \mathbf{\Phi}_1\,\mathbf{D}\,\mathbf{\Phi}_1' + ... + \mathbf{\Phi}_{h-1}\,\mathbf{D}\,\mathbf{\Phi}_{h-1}'$$

The elements on the main diagonal capture the forecast error variances of each variable in $\mathbf{y}$. Writing the matrix in full form we get:

$$E\left[\begin{pmatrix} y_{1,t+h} - E\left(y_{1,t+h}\left|\mathbf{y}_t, \mathbf{y}_{t-1}, ...\right.\right) \\ y_{2,t+h} - E\left(y_{2,t+h}\left|\mathbf{y}_t, \mathbf{y}_{t-1}, ...\right.\right) \end{pmatrix}\begin{pmatrix} y_{1,t+h} - E\left(y_{1,t+h}\left|\mathbf{y}_t, \mathbf{y}_{t-1}, ...\right.\right) \\ y_{2,t+h} - E\left(y_{2,t+h}\left|\mathbf{y}_t, \mathbf{y}_{t-1}, ...\right.\right) \end{pmatrix}'\right] \equiv$$

$$\equiv \begin{pmatrix} var(y_{1,t+h} - E\left(y_{1,t+h}\left|\mathbf{y}_t, \mathbf{y}_{t-1}, ...\right.\right)) & cov(.,.) \\ cov(.,.) & var(y_{2,t+h} - E\left(y_{2,t+h}\left|\mathbf{y}_t, \mathbf{y}_{t-1}, ...\right.\right)) \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} \phi_{0,11} & \phi_{0,12} \\ \phi_{0,21} & \phi_{0,22} \end{pmatrix}}_{\mathbf{\Phi}_0}\underbrace{\begin{pmatrix} \sigma^2_{v_1} & 0 \\ 0 & \sigma^2_{v_2} \end{pmatrix}}_{\mathbf{D}}\underbrace{\begin{pmatrix} \phi_{0,11} & \phi_{0,21} \\ \phi_{0,12} & \phi_{0,22} \end{pmatrix}}_{\mathbf{\Phi}_0'} +$$

$$+ \underbrace{\begin{pmatrix} \phi_{1,11} & \phi_{1,12} \\ \phi_{1,21} & \phi_{1,22} \end{pmatrix}}_{\mathbf{\Phi}_1}\underbrace{\begin{pmatrix} \sigma^2_{v_1} & 0 \\ 0 & \sigma^2_{v_2} \end{pmatrix}}_{\mathbf{D}}\underbrace{\begin{pmatrix} \phi_{1,11} & \phi_{1,21} \\ \phi_{1,12} & \phi_{1,22} \end{pmatrix}}_{\mathbf{\Phi}_1'} + ...$$

$$... + \underbrace{\begin{pmatrix} \phi_{h-1,11} & \phi_{h-1,12} \\ \phi_{h-1,21} & \phi_{h-1,22} \end{pmatrix}}_{\mathbf{\Phi}_{h-1}}\underbrace{\begin{pmatrix} \sigma^2_{v_1} & 0 \\ 0 & \sigma^2_{v_2} \end{pmatrix}}_{\mathbf{D}}\underbrace{\begin{pmatrix} \phi_{h-1,11} & \phi_{h-1,21} \\ \phi_{h-1,12} & \phi_{h-1,22} \end{pmatrix}}_{\mathbf{\Phi}_{h-1}'} =$$

(omitting off-diagonal covariance terms for ease of exposition)

$$= \underbrace{\begin{pmatrix} \sigma^2_{v_1}\phi^2_{0,11} + \sigma^2_{v_2}\phi^2_{0,12} & ... \\ ... & \sigma^2_{v_1}\phi^2_{0,21} + \sigma^2_{v_2}\phi^2_{0,22} \end{pmatrix}}_{\mathbf{\Phi}_0\,\mathbf{D}\,\mathbf{\Phi}_0'} +$$

$$+ \underbrace{\begin{pmatrix} \sigma^2_{v_1}\phi^2_{1,11} + \sigma^2_{v_2}\phi^2_{1,12} & ... \\ ... & \sigma^2_{v_1}\phi^2_{1,21} + \sigma^2_{v_2}\phi^2_{1,22} \end{pmatrix}}_{\mathbf{\Phi}_1\,\mathbf{D}\,\mathbf{\Phi}_1'} + ...$$

$$... + \underbrace{\begin{pmatrix} \sigma^2_{v_1}\phi^2_{h-1,11} + \sigma^2_{v_2}\phi^2_{h-1,12} & ... \\ ... & \sigma^2_{v_1}\phi^2_{h-1,21} + \sigma^2_{v_2}\phi^2_{h-1,22} \end{pmatrix}}_{\mathbf{\Phi}_{h-1}\,\mathbf{D}\,\mathbf{\Phi}_{h-1}'} =$$

$$= \begin{pmatrix} \sigma^2_{v_1}\sum_{s=0}^{h-1}\phi^2_{s,11} + \sigma^2_{v_2}\sum_{s=0}^{h-1}\phi^2_{s,12} & ... \\ ... & \sigma^2_{v_1}\sum_{s=0}^{h-1}\phi^2_{s,21} + \sigma^2_{v_2}\sum_{s=0}^{h-1}\phi^2_{s,22} \end{pmatrix}$$

At any horizon h, the forecast error variance for each variable i (with $i = 1, 2$) is the sum of two components: (i) $\sigma^2_{v_1} \sum_{s=0}^{h-1} \phi^2_{s,i1}$, that captures the variance due to the first structural disturbance v_1, and (ii) $\sigma^2_{v_2} \sum_{s=0}^{h-1} \phi^2_{s,i2}$, capturing the variance due to v_2. The $FEVD$ exercise consists in computing the fractions of the total variance attributable to each of the two structural shocks. Letting $FEVD_{h,ij}$ denote the portion of the forecast error variance of variable i attributable to shock j at forecasting horizon h, we have:

• for a one-period forecasting horizon, $h = 1$:

$$FEVD_{1,11} = \frac{\sigma^2_{v_1}\phi^2_{0,11}}{\sigma^2_{v_1}\phi^2_{0,11} + \sigma^2_{v_2}\phi^2_{0,12}}, \; FEVD_{1,12} = \frac{\sigma^2_{v_2}\phi^2_{0,12}}{\sigma^2_{v_1}\phi^2_{0,11} + \sigma^2_{v_2}\phi^2_{0,12}} \quad \text{for } y_1$$

$$FEVD_{1,21} = \frac{\sigma^2_{v_1}\phi^2_{0,21}}{\sigma^2_{v_1}\phi^2_{0,21} + \sigma^2_{v_2}\phi^2_{0,22}}, \; FEVD_{1,22} = \frac{\sigma^2_{v_2}\phi^2_{0,22}}{\sigma^2_{v_1}\phi^2_{0,21} + \sigma^2_{v_2}\phi^2_{0,22}} \quad \text{for } y_2$$

• for a two-period forecasting horizon, $h = 2$:

$$FEVD_{1,11} = \frac{\sigma^2_{v_1}\sum_{s=0}^{1}\phi^2_{s,11}}{\sigma^2_{v_1}\sum_{s=0}^{1}\phi^2_{s,11} + \sigma^2_{v_2}\sum_{s=0}^{1}\phi^2_{s,12}}, \; FEVD_{1,12} = \frac{\sigma^2_{v_2}\sum_{s=0}^{1}\phi^2_{s,12}}{\sigma^2_{v_1}\sum_{s=0}^{1}\phi^2_{s,11} + \sigma^2_{v_2}\sum_{s=0}^{1}\phi^2_{s,12}} \quad \text{for } y_1$$

$$FEVD_{1,21} = \frac{\sigma^2_{v_1}\sum_{s=0}^{1}\phi^2_{s,21}}{\sigma^2_{v_1}\sum_{s=0}^{1}\phi^2_{s,21} + \sigma^2_{v_2}\sum_{s=0}^{1}\phi^2_{s,22}}, \; FEVD_{1,22} = \frac{\sigma^2_{v_2}\sum_{s=0}^{1}\phi^2_{s,22}}{\sigma^2_{v_1}\sum_{s=0}^{1}\phi^2_{s,21} + \sigma^2_{v_2}\sum_{s=0}^{1}\phi^2_{s,22}} \quad \text{for } y_2$$

and so on for more extended forecasting horizons. In general, for horizon h, variable i and shock j, we have:

$$FEVD_{h,ij} = \frac{\sigma^2_{v_j}\sum_{s=0}^{h-1}\phi^2_{s,ij}}{\sigma^2_{v_1}\sum_{s=0}^{h-1}\phi^2_{s,i1} + \sigma^2_{v_2}\sum_{s=0}^{h-1}\phi^2_{s,i2}}$$