Basics: spot rates, implicit forward rates and the yield curve

Simple case (discrete time): *zero-coupon* bonds (pay 1 at maturity), no *default risk*.

Yield to maturity (spot):

$$P_T = \frac{1}{(1+R_T)^T}$$

T: maturity (in periods, e.g. years)

P_T: price at time 0 of a bond maturing at T

R_T: *yield to maturity* of the bond

The relation between yields to maturity R_T to maturity dates T is known as "term structure of interest rates" (*yield curve*)

Yields to maturity R_T *implicitly* define a term structure of one-period *forward* interest rates

In general, a (one-period) *forward* contract signed at 0 fixes the payment obtainable at T by investing 1 unit of capital at $T-1$.

Equivalently (as a result of two *spot* contracts):

	0	$T-1$	T
purchase bond maturing at T	-1		$(1+R_T)^T$
sell bond maturing at $T-1$	$+1$	$-(1+R_{T-1})^{T-1}$	
	0	$-(1+R_{T-1})^{T-1}$	$(1+R_T)^T$

⇒ the ***forward*** rate between $T-1$ and T, denoted as ${}_{T-1}f_T$, is

$$1 + {}_{T-1}f_T = \frac{(1+R_T)^T}{(1+R_{T-1})^{T-1}}$$

from which:

$${}_{T-1}f_T = \frac{(1+R_T)^T - (1+R_{T-1})^{T-1}}{(1+R_{T-1})^{T-1}}$$

the *forward* rate ${}_{T-1}f_T$ is the increase (in percentage points) of the final sum obtainable from a unitary investment at time 0, if the maturity is delayed by one period, from $T-1$ to T.

Equivalently, in terms of the **price** of the bond at 0 :

$$1 +_{T-1} f_T = \frac{P_{T-1}}{P_T}$$

from which:

$$_{T-1}f_T = \frac{P_{T-1} - P_T}{P_T}$$

the *forward* rate $_{T-1}f_T$ is the increase (in percentage points) of the price of the bond at time 0, if the maturity is anticipated by one period, from T to $T-1$.

The same relationship between *spot* and (one-period) *forward* rates holds for any pair of dates. Therefore (going backward in time from T):

$$(1+R_T)^T = (1+R_{T-1})^{T-1} \cdot (1+{}_{T-1}f_T)$$

$$(1+R_{T-1})^{T-1} = (1+R_{T-2})^{T-2} \cdot (1+{}_{T-2}f_{T-1})$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$(1+R_2)^2 = (1+R_1) \cdot (1+{}_1f_2)$$

Combining the above expressions (with $R_1 \equiv {}_0 f_1$):

$$(1+R_T)^T = (1+{}_0 f_1) \bullet (1+{}_1 f_2) \ldots (1+{}_{T-1} f_T)$$

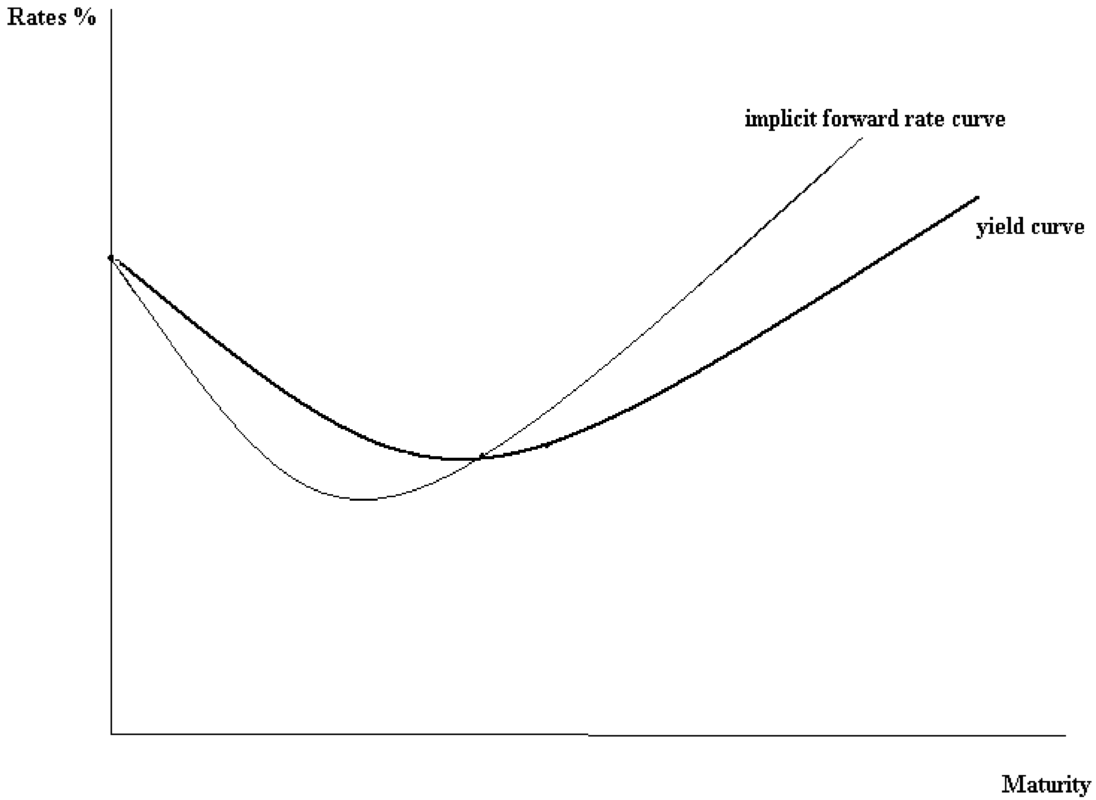
⇒ link between the *spot* rate on a bond maturing at T and the series of (one-period) *forward* rates from period 1 to T.

Using the logarithmic approximation $\log(1+x) \simeq x$ we get:

$$R_T = \frac{1}{T}\left({}_0 f_1 + {}_1 f_2 + {}_2 f_3 + \ldots + {}_{T-1} f_T\right)$$

⇒ for any maturity T the corresponding *yield* R_T can be expressed as the average of the implicit *forward* rates on the whole life of the bond

⇒ An **implicit forward rate curve** is therefore associated to the **yield** (to maturity) **curve**



Note: Relationship between the implicit *forward rate* curve and the *yield* curve in:

$$\underbrace{{}_{T-1}f_T}_{\text{implicit forward rate between } T-1 \text{ and } T} = \underbrace{R_{T-1}}_{\text{spot rate at } T-1} + \underbrace{(R_T - R_{T-1})}_{\text{"slope" of the yield curve between } T-1 \text{ and } T} \cdot T$$

Hence:

${}_{T-1}f_T > R_{T-1}$ if *spot* rates increase b/w $T-1$ and T

${}_{T-1}f_T < R_{T-1}$ if *spot* rates decrease b/w $T-1$ and T

${}_{T-1}f_T = R_{T-1}$ for $T = 0$ or if *spot* rates are constant b/w $T-1$ and T