Monetary Economics 2
Notes on VAR modelling for monetary policy analysis

F.C. Bagliano

September 2017

1 Introduction to VAR modelling

Univariate autoregressive (AR) models can be extended to the multivariate case to study dynamic interrelationships among several variables, all viewed as *endogenous*. The resulting *vector autoregression* (VAR) models describe the evolution over time of a vector of n variables $\mathbf{y}_t = (y_{1t}\ y_{2t}...y_{nt})'$ as a function of its past realizations $\mathbf{y}_{t-1}, \mathbf{y}_{t-2}, ...$ and a vector of stochastic terms $\mathbf{u}_t = (u_{1t}\ u_{2t}...u_{nt})'$. A VAR model with p lags of the endogenous variables in $\mathbf{y}_t$ is called $VAR(p)$ and has the following general form:

$$\mathbf{y}_t = \boldsymbol{\delta} + \boldsymbol{\Theta}_1\,\mathbf{y}_{t-1} + \boldsymbol{\Theta}_2\,\mathbf{y}_{t-2} + ... + \boldsymbol{\Theta}_p\,\mathbf{y}_{t-p} + \mathbf{u}_t$$

where $\boldsymbol{\delta} = (\delta_1\ \delta_2...\delta_n)'$ is a vector of constant terms, $\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2...\boldsymbol{\Theta}_p$ are $n \times n$ matrices and $\mathbf{u}_t$ is a vector of *white noise* processes with the following variance-covariance structure:

$$E(\mathbf{u}_t\mathbf{u}_s') = \begin{cases} \boldsymbol{\Sigma} & \text{if } t = s \\ \mathbf{0} & \text{if } t \neq s \end{cases}$$

Matrix $\boldsymbol{\Sigma}$ is not assumed to be diagonal: therefore the error terms of the individual equations can be (contemporaneously) correlated.

To understand the nature of the contemporaneous correlation among the elements of $\mathbf{u}_t$ it is useful to view the VAR model as the "**reduced form**" of a "**structural**" model capturing behavioral relationships among the endogenous variables. As an example, in the case of two variables with dynamics limited to one lag only, the structural form of the model is given by

$$\begin{aligned} y_{1t} &= \gamma_{10} + a_{12}\,y_{2t} + \gamma_{11}\,y_{1t-1} + \gamma_{12}\,y_{2t-1} + v_{1t} \\ y_{2t} &= \gamma_{20} + a_{21}\,y_{1t} + \gamma_{21}\,y_{1t-1} + \gamma_{22}\,y_{2t-1} + v_{2t} \end{aligned}$$

with

$$E(\mathbf{v}_t\mathbf{v}_s') = \begin{cases} \mathbf{D} & \text{if } t = s \quad \text{with } \mathbf{D} \text{ diagonal} \\ \mathbf{0} & \text{if } t \neq s \end{cases}$$

The elements of $\mathbf{v}_t$ are the *structural shocks* (or disturbances) of the system. The fact that they are *uncorrelated* (since $\mathbf{D}$ is diagonal, implying $E(v_{1t}v_{2t}) = 0$), allows for a precise economic interpretation of the structural disturbances in terms, for example, of demand shocks, supply shocks, innovations to monetary policy, etc. Indeed, one of the main purposes of VAR modelling is the estimation of the dynamic response of all endogenous variables to structural disturbances hitting the system. In matrix notation, the structural form of the system can be expressed as:

$$\begin{aligned} \underbrace{\begin{pmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} &= \underbrace{\begin{pmatrix} \gamma_{10} \\ \gamma_{20} \end{pmatrix}}_{\boldsymbol{\gamma}} + \underbrace{\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}}_{\boldsymbol{\Gamma}} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \underbrace{\begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}}_{\mathbf{v}} \\ \mathbf{A}\,\mathbf{y}_t &= \boldsymbol{\gamma} + \boldsymbol{\Gamma}\,\mathbf{y}_{t-1} + \mathbf{v}_t \end{aligned}$$

Matrix $\mathbf{A}$ captures the contemporaneous relations between the two endogenous variables. Inverting $\mathbf{A}$ we get the reduced form of the model as a $VAR(1)$:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_{10} \\ \gamma_{20} \end{pmatrix} + \begin{pmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix}$$
$$+ \begin{pmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{pmatrix}^{-1} \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}$$

$$\begin{aligned} \mathbf{y}_t &= \mathbf{A}^{-1}\boldsymbol{\gamma} + \mathbf{A}^{-1}\Gamma\,\mathbf{y}_{t-1} + \mathbf{A}^{-1}\mathbf{v}_t \\ \Rightarrow \quad \mathbf{y}_t &= \boldsymbol{\delta} + \boldsymbol{\Theta}\,\mathbf{y}_{t-1} + \mathbf{u}_t \end{aligned}$$

where the variance-covariance matrix of the error terms $\boldsymbol{\Sigma}$ is given by

$$\boldsymbol{\Sigma} = \frac{1}{(1 - a_{12}a_{21})^2} \begin{pmatrix} \sigma_{v_1}^2 + a_{12}^2\sigma_{v_2}^2 & a_{21}\sigma_{v_1}^2 + a_{12}\sigma_{v_2}^2 \\ a_{21}\sigma_{v_1}^2 + a_{12}\sigma_{v_2}^2 & a_{21}^2\sigma_{v_1}^2 + \sigma_{v_2}^2 \end{pmatrix}$$

The VAR error terms (elements of $\mathbf{u}_t$), interpreted as one-step forecast errors or "innovations", are linear combinations of the structural disturbances v_{1t} and v_{2t} and, in general, have non-zero covariance.

1.1 Stationarity

Using the VAR representation, it is possible to study the dynamic response of the variables in $\mathbf{y}_t$ to each element of vector $\mathbf{u}_t$. If the VAR is *stationary* (that is, it has finite and time-invariant first and second moments), the response to innovations gradually dies down, tending to zero in the long-run. It is therefore important to establish under what conditions the VAR system is stationary. Such conditions are a multivariate extension of those valid for the unvariate AR process.

In the $VAR(p)$ case

$$\begin{gathered}\mathbf{y}_t = \boldsymbol{\delta} + \boldsymbol{\Theta}_1\, \mathbf{y}_{t-1} + \boldsymbol{\Theta}_2\, \mathbf{y}_{t-2} + \dots + \boldsymbol{\Theta}_p\, \mathbf{y}_{t-p} + \mathbf{u}_t \\ (\mathbf{I} - \boldsymbol{\Theta}_1\, L - \boldsymbol{\Theta}_2\, L^2 - \dots - \boldsymbol{\Theta}_p\, L^p)\, \mathbf{y}_t = \boldsymbol{\delta} + \mathbf{u}_t \\ \boldsymbol{\Theta}(L)\, \mathbf{y}_t = \boldsymbol{\delta} + \mathbf{u}_t\end{gathered}$$

for $\mathbf{y}_t$ to be stationary, the matrix polynomial in the lag operator $\boldsymbol{\Theta}(L)$ must be invertible. Invertibility requires that the roots of the following equation

$$\det\ (\mathbf{I} - \boldsymbol{\Theta}_1\, z - \boldsymbol{\Theta}_2\, z^2 - \dots - \boldsymbol{\Theta}_p\, z^p) = 0$$

be (in modulus) outside the unit circle.

Example: for a $VAR(1)$ bivariate system

$$\begin{gathered}\mathbf{y}_t = \boldsymbol{\delta} + \boldsymbol{\Theta}_1\, \mathbf{y}_{t-1} + \mathbf{u}_t \\ \Rightarrow\ (\mathbf{I} - \boldsymbol{\Theta}_1\, L)\, \mathbf{y}_t = \boldsymbol{\delta} + \mathbf{u}_t \\ \begin{pmatrix} 1-\theta_{11}L & -\theta_{12}L \\ -\theta_{21}L & 1-\theta_{22}L \end{pmatrix} \mathbf{y}_t = \boldsymbol{\delta} + \mathbf{u}_t\end{gathered}$$

the stationarity condition is that the roots of the equation

$$\begin{gathered}\det\ (\mathbf{I} - \boldsymbol{\Theta}_1\, z) = 0 \\ \Rightarrow\ 1 - (\theta_{11} + \theta_{22})\, z + (\theta_{11}\theta_{22} - \theta_{21}\theta_{12})\, z^2 = 0\end{gathered}$$

be outside the unit circle.

Under stationarity, the VAR can be expressed as a *vector moving average* of infinite order, $VMA(\infty)$:

$$\begin{aligned}\mathbf{y}_t &= \boldsymbol{\Theta}(L)^{-1}\, (\boldsymbol{\delta} + \mathbf{u}_t) \\ &= \boldsymbol{\Theta}(1)^{-1}\, \boldsymbol{\delta} + \boldsymbol{\Theta}(L)^{-1}\, \mathbf{u}_t\end{aligned}$$

from which we get the *expected value* (unconditional mean) of $\mathbf{y}_t$:

$$E(\mathbf{y}_t) = \boldsymbol{\Theta}(1)^{-1}\, \boldsymbol{\delta} = (\mathbf{I} - \boldsymbol{\Theta}_1 - \boldsymbol{\Theta}_2 - \dots - \boldsymbol{\Theta}_p)^{-1}\, \boldsymbol{\delta}$$

1.2 Estimation

The VAR can be estimated by means of OLS applied equation by equation. This procedure yields consistent estimates of the parameters in $\boldsymbol{\delta}$, $\boldsymbol{\Theta}_1$, $\boldsymbol{\Theta}_2$, ... $\boldsymbol{\Theta}_p$, and of the variances and covariances of the innovations in matrix $\boldsymbol{\Sigma}$.

1.3 The "identification" problem and dynamic analysis

If we are interested in analyzing the dynamic response of the endogenous variables in $\mathbf{y}_t$ to each element of the innovation vector $\mathbf{u}_t$, the $VMA(\infty)$ form of the VAR model can be used:

$$\begin{aligned} \mathbf{y}_t &= \boldsymbol{\Theta}(1)^{-1}\,\boldsymbol{\delta} + \boldsymbol{\Theta}(L)^{-1}\,\mathbf{u}_t \\ &= \boldsymbol{\Theta}(1)^{-1}\,\boldsymbol{\delta} + (\mathbf{I} + \boldsymbol{\Psi}_1 L + \boldsymbol{\Psi}_2 L^2 + ...)\,\mathbf{u}_t \end{aligned}$$

The $n \times n$ matrices $\boldsymbol{\Psi}_s$ (for $s = 0, ...\infty$ and with $\boldsymbol{\Psi}_0 = \mathbf{I}$) describe the response over time of each endogenous variable y_i $(i = 1, ...n)$ to each shock u_j $(j = 1, ...n)$.

However, the economic interpretability of such responses is often difficult since the VAR innovations (elements of $\mathbf{u}_t$) are linear combinations of the system's structural disturbances and display correlation. Indeed, what we would like to get from the dynamic analysis of the VAR model is the response of $\mathbf{y}_t$ to the structural disturbances, to which it is generally possible to attach an economic interpretation. Extending the above example (with $n = 2$ and $p = 1$) to $n > 2$ endogenous variables and to $p > 1$ lags, we can write the relationship between the structural and reduced forms of the system (and the associated disturbances) as:

$$\begin{aligned} \mathbf{A}\,\mathbf{y}_t &= \boldsymbol{\gamma} + \boldsymbol{\Gamma}_1\,\mathbf{y}_{t-1} + \boldsymbol{\Gamma}_2\,\mathbf{y}_{t-2} + ... + \boldsymbol{\Gamma}_p\,\mathbf{y}_{t-p} + \mathbf{v}_t \\ \Rightarrow\quad \mathbf{y}_t &= \mathbf{A}^{-1}\boldsymbol{\gamma} + \mathbf{A}^{-1}\boldsymbol{\Gamma}_1\,\mathbf{y}_{t-1} + \mathbf{A}^{-1}\boldsymbol{\Gamma}_2\,\mathbf{y}_{t-2} + ... + \mathbf{A}^{-1}\boldsymbol{\Gamma}_p\,\mathbf{y}_{t-p} + \mathbf{A}^{-1}\mathbf{v}_t \\ &= \boldsymbol{\delta} + \boldsymbol{\Theta}_1\,\mathbf{y}_{t-1} + \boldsymbol{\Theta}_2\,\mathbf{y}_{t-2} + ... + \boldsymbol{\Theta}_p\,\mathbf{y}_{t-p} + \mathbf{u}_t \end{aligned}$$

from which we get the relationship between the VAR residuals ($\mathbf{u}_t$) and the structural disturbances ($\mathbf{v}_t$):

$$\begin{aligned} \mathbf{u}_t &= \mathbf{A}^{-1}\mathbf{v}_t \\ \Rightarrow\quad \mathbf{A}\,\mathbf{u}_t &= \mathbf{v}_t \end{aligned}$$

To obtain the structural disturbances $\mathbf{v}_t$ from estimation of the VAR's innovations $\mathbf{u}_t$, it is necessary to identify the elements of matrix $\mathbf{A}$ (containing the contemporaneous relationships among the endogenous variables). Only after solving this identification problem it is possible to proceed to the analysis of the dynamic response of $\mathbf{y}_t$ to each shock in $\mathbf{v}_t$. To obtain the elements of $\mathbf{A}$ we can use the relation between the variance-covariance matrices of $\mathbf{u}_t$ ($\mathbf{\Sigma}$) and $\mathbf{v}_t$ ($\mathbf{D}$), the latter being a diagonal matrix:

$$\begin{aligned} E(\mathbf{A}\,\mathbf{u}_t\,(\mathbf{A}\,\mathbf{u}_t)') &= \mathbf{A}\,\underbrace{E(\mathbf{u}_t\mathbf{u}_t')}_{\mathbf{\Sigma}}\,\mathbf{A}' = \underbrace{E(\mathbf{v}_t\mathbf{v}_t')}_{\mathbf{D}} \\ &\Rightarrow \quad \mathbf{A}\,\mathbf{\Sigma}\,\mathbf{A}' = \mathbf{D} \end{aligned}$$

The parameters to be identified are the $n(n-1)$ off-diagonal elements of $\mathbf{A}$ (by construction the elements on the main diagonal being set equal to 1), and the n elements on the diagonal of $\mathbf{D}$ (the variances of the structural shocks): overall, n^2 parameters. Estimation of the VAR yields estimates of n variances and $\frac{n(n-1)}{2}$ covariances among the innovations: overall, $\frac{n(n+1)}{2}$ estimates, corresponding to the distinct elements of the symmetric matrix $\mathbf{\Sigma}$. Since the number of available estimates is less than the number of parameters to be identified, it is necessary to impose a set of $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$ restrictions on the elements of $\mathbf{A}$. Such restrictions, concerning the simultaneous relationships among the endogenous variables, can be suggested by economic theory.

Example (for $n = 2$). In the bivariate case, the relationship between the VAR residuals and the structural disturbances is:

$$\begin{pmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} = \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}$$

There are 4 parameters to be identified: two off-diagonal elements of $\mathbf{A}$ ($-a_{12}$ and $-a_{21}$), and the two variances of the structural shocks $\sigma^2_{v_1}$ and $\sigma^2_{v_2}$. From VAR estimation we get estimates s_{ij} (with $i, j = 1, 2$) of the elements of $\mathbf{\Sigma}$ (with $s_{12} = s_{21}$). The relationship between $\mathbf{\Sigma}$ and $\mathbf{D}$ is then:

$$\begin{pmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} 1 & -a_{21} \\ -a_{12} & 1 \end{pmatrix} = \begin{pmatrix} \sigma^2_{v_1} & 0 \\ 0 & \sigma^2_{v_2} \end{pmatrix}$$

from which we get the following system of 3 equations:

$$\begin{aligned} s_{11} - 2a_{12}s_{12} + a_{12}^2 s_{22} &= \sigma^2_{v_1} \\ a_{21}^2 s_{11} - 2a_{21}s_{12} + s_{22} &= \sigma^2_{v_2} \\ -a_{21}s_{11} + (1 + a_{21}a_{12})s_{12} - a_{12}s_{22} &= 0 \end{aligned}$$

It is necessary to impose one restriction to achieve identification. For example, if the chosen restriction is $a_{12} = 0$, the system becomes:

$$\begin{aligned} s_{11} &= \sigma_{v_1}^2 \\ a_{21}^2 s_{11} - 2a_{21}s_{12} + s_{22} &= \sigma_{v_2}^2 \\ -a_{21}s_{11} + s_{12} &= 0 \end{aligned}$$

from which we get the 3 remaining parameters:

$$\begin{aligned} \sigma_{v_1}^2 &= s_{11} \\ a_{21} &= \frac{s_{12}}{s_{11}} \\ \sigma_{v_2}^2 &= s_{22} - \frac{s_{12}^2}{s_{11}} \end{aligned}$$

This is the simplest example of "*triangular*" (or "*Choleski*") identification, which imposes a recursive structure to the system of simultaneous relations among the variables. In this case the structural shock v_{1t} is immediately identified as the first element of the vector of VAR residuals (u_{1t}), and the shock v_{2t} is identified as that portion of u_{2t} not correlated with (i.e. "orthogonal" to) u_{1t}:

$$\begin{aligned} u_{1t} &= v_{1t} \\ u_{2t} &= a_{21}u_{1t} + v_{2t} \end{aligned}$$

The chosen restriction for identification ($a_{12} = 0$) corresponds to a specific "ordering" of the variables in the system (with y_1 coming "before" y_2): y_1 affects y_2 within period t whereas y_2 does affect y_1 only in subsequent periods (through its lags $y_{2t-1,\ldots}$ in the VAR equation for y_{1t}). The alternative restriction $a_{21} = 0$ leads to a different identification, based on the opposite ordering of the variables, with y_2 before y_1; in this case we have

$$\begin{aligned} u_{2t} &= v_{2t} \\ u_{1t} &= a_{12}u_{2t} + v_{1t} \end{aligned}$$

In general, simple recursive identification schemes of this kind make use of the Choleski factorization of a symmetric, positive definite matrix such as $\mathbf{\Sigma}$.[1]

[1] *Choleski factorization*: any symmetric positive definite matrix $\mathbf{\Sigma}$ can be expressed

After achieving identification of the parameters in $\mathbf{A}$ and $\mathbf{D}$, the (orthogonal) structural disturbances can be constructed from the relation $\mathbf{A}\,\mathbf{u}_t = \mathbf{v}_t$. It is then possible to proceed to the analysis of the dynamic response of $\mathbf{y}_{t+i}$ to $\mathbf{v}_t$. This analysis, called **innovation accounting**, is carried out by constructing the (orthogonalized) *impulse response functions* (IRF) and *forecast errors variance decompositions* ($FEVD$), both derived from the $VMA(\infty)$ representation of the VAR system:

$$\begin{aligned}
\mathbf{y}_t &= \mathbf{\Theta}(1)^{-1}\,\boldsymbol{\delta} + (\mathbf{I} + \mathbf{\Psi}_1 L + \mathbf{\Psi}_2 L^2 + ...)\,\mathbf{u}_t \\
&= \mathbf{\Theta}(1)^{-1}\,\boldsymbol{\delta} + (\underbrace{\mathbf{A}^{-1}}_{\mathbf{\Phi}_0} + \underbrace{\mathbf{\Psi}_1\mathbf{A}^{-1}}_{\mathbf{\Phi}_1} L + \underbrace{\mathbf{\Psi}_2\mathbf{A}^{-1}}_{\mathbf{\Phi}_2} L^2 + ...)\,\mathbf{v}_t \\
&= \mathbf{\Theta}(1)^{-1}\,\boldsymbol{\delta} + \mathbf{\Phi}_0\,\mathbf{v}_t + \mathbf{\Phi}_1\,\mathbf{v}_{t-1} + \mathbf{\Phi}_2\,\mathbf{v}_{t-2} + ...
\end{aligned}$$

IRF: the elements of the matrices $\mathbf{\Phi}_i$ trace out the effects over time (*impulse response functions*) of each structural disturbance in $\mathbf{v}$ keeping all the other disturbances at zero, under the set of identifying assumptions used in the preceding step (note that $\mathbf{\Phi}_0 = \mathbf{A}^{-1}$). Each matrix $\mathbf{\Phi}_s$ captures the effect of the structural shocks at time t on the endogenous variables at time $t + s$; the typical element

$$\phi_{s,ij} = \frac{\partial\, y_{i,t+s}}{\partial\, v_{j,t}}$$

captures the response of the ith element of $\mathbf{y}_{t+s}$ to an "impulse" due to the jth element of $\mathbf{v}_t$.

$FEVD$: from the $VMA(\infty)$ representation of the VAR it is possible to obtain the forecast of future $\mathbf{y}'$s over an h-period horizon ($\hat{\mathbf{y}}_{t+h}$) on the basis of information in current (time t) and past values of the variables in the system $\mathbf{y}_t, \mathbf{y}_{t-1}, ...$ The associated forecast error is:

$$\mathbf{y}_{t+h} - E\left(\mathbf{y}_{t+h}\,|\mathbf{y}_t, \mathbf{y}_{t-1}, ...\right) = \mathbf{\Phi}_0\,\mathbf{v}_{t+h} + \mathbf{\Phi}_1\,\mathbf{v}_{t+h-1} + ... + \mathbf{\Phi}_{h-1}\,\mathbf{v}_{t+1}$$

("*factorized*") as

$$\mathbf{\Sigma} = \mathbf{T}\mathbf{T}'$$

with $\mathbf{T}$ a lower-triangular matrix (with all zeros above the main diagonal). In the case above

$$\mathbf{T} = \mathbf{A}^{-1}\mathbf{D}^{\frac{1}{2}}$$

so that

$$\begin{aligned}
\mathbf{\Sigma} &= (\mathbf{A}^{-1}\mathbf{D}^{\frac{1}{2}})(\mathbf{A}^{-1}\mathbf{D}^{\frac{1}{2}})' \\
&= \mathbf{A}^{-1}\mathbf{D}(\mathbf{A}^{-1})'
\end{aligned}$$

The *forecast error variance* is given by the following symmetric matrix

$$var\ (\mathbf{y}_{t+h} - E\ (\mathbf{y}_{t+h}\ |\mathbf{y}_t, \mathbf{y}_{t-1}, ...)) = \mathbf{\Phi}_0\, \mathbf{D}\, \mathbf{\Phi}_0' + \mathbf{\Phi}_1\, \mathbf{D}\, \mathbf{\Phi}_1' + ... + \mathbf{\Phi}_{h-1}\, \mathbf{D}\, \mathbf{\Phi}_{h-1}'$$

The elements on the main diagonal of this matrix capture the forecast error variances of each variable in $\mathbf{y}$: $var\ (y_{i,t+h} - E(y_{i,t+h}\ |\mathbf{y}_t, \mathbf{y}_{t-1}, ...))$. This variance can be expressed as the sum of the contributions of each structural disturbance in $\mathbf{v}$ to the total variance of each endogenous variable over the relevant horizon (*forecast error variance decomposition*). For example, the fraction of the forecast error variance of the i-th endogenous variable at horizon h attributable to the j-th structural disturbance is given by

$$\frac{\sigma_{v_j}^2 \sum_{s=0}^{h-1} \phi_{s,ij}^2}{\sigma_{v_1}^2 \sum_{s=0}^{h-1} \phi_{s,i1}^2 + \sigma_{v_2}^2 \sum_{s=0}^{h-1} \phi_{s,i2}^2 + ... + \sigma_{v_n}^2 \sum_{s=0}^{h-1} \phi_{s,in}^2}$$

Again (as for the IRFs), this step is economically meaningful only if there are no covariance terms, which is warranted by the chosen identification procedure ($\mathbf{D}$ is diagonal).

1.4 A first application: Stock-Watson (*JEP* 2001)

J. Stock and M. Watson (*Journal of Economic Perspectives* 2001) provide a simple example of VAR modelling, studying a small-scale macroeconomic system for the US made up of the inflation rate (π), the unemployment rate (U) and the Federal Funds rate (R), the very short-term policy rate directly influenced by the monetary policy decisions of the Federal Reserve. The structural form of the system is then:

$$\mathbf{A} \begin{pmatrix} \pi_t \\ U_t \\ R_t \end{pmatrix} = \mathbf{C}(L) \begin{pmatrix} \pi_{t-1} \\ U_{t-1} \\ R_{t-1} \end{pmatrix} + \begin{pmatrix} v_t^1 \\ v_t^2 \\ v_t^P \end{pmatrix}$$

where $\mathbf{C}(L)$ is a square matrix of polynomials in the lag operator. The innovation accounting analysis is performed after identification of the structural disturbances achieved by means of a Choleski (recursive) factorization of matrix $\mathbf{A}$. Ordering the three variables as shown above, the identification scheme can be represented as:

$$\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} \begin{pmatrix} u_t^\pi \\ u_t^U \\ u_t^R \end{pmatrix} = \begin{pmatrix} v_t^1 \\ v_t^2 \\ v_t^P \end{pmatrix}$$

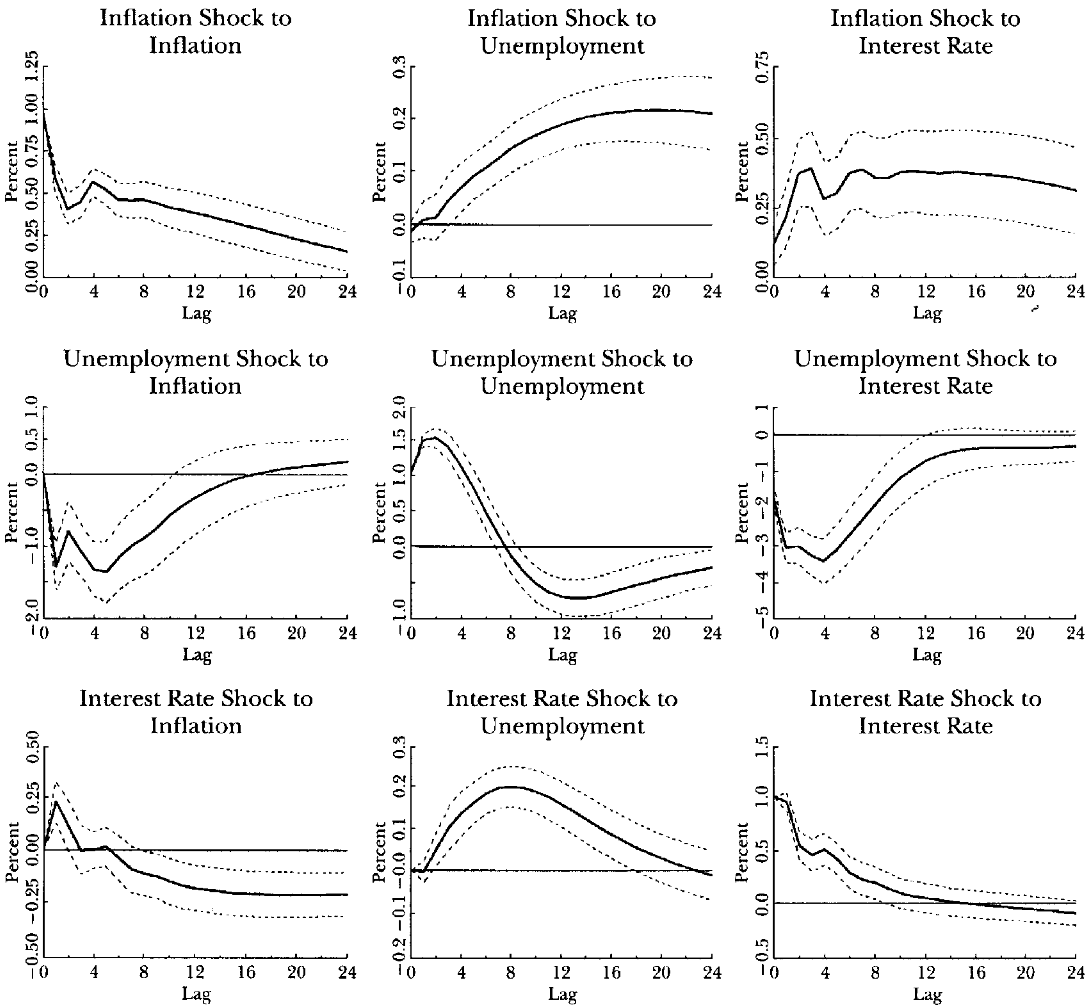
$$\Rightarrow \begin{array}{l} u_t^\pi = v_t^1 \\ u_t^U = -a_{21} u_t^\pi + v_t^2 \\ u_t^R = -a_{31} u_t^\pi - a_{32} u_t^U + v_t^P \end{array}$$

where the structural disturbance v_t^P is interpreted as an unanticipated movement of the interest rate due to monetary policy actions. Inverting $\mathbf{A}$ yields the VAR innovations as linear combinations of the structural disturbances:

$$
\begin{aligned}
u_t^\pi &= v_t^1 \\
u_t^U &= -a_{21} v_t^1 + v_t^2 \\
u_t^R &= -(a_{31} - a_{32} a_{21})\, v_t^1 - a_{32}\, v_t^2 + v_t^P
\end{aligned}
$$

Figure 1

Impulse Responses in the Inflation-Unemployment-Interest Rate Recursive VAR



B. *Variance Decompositions from the Recursive VAR Ordered as π, u, R*

B.i. Variance Decomposition of π

Forecast Horizon	*Forecast Standard Error*	*Variance Decomposition (Percentage Points)* π	u	R
1	0.96	100	0	0
4	1.34	88	10	2
8	1.75	82	17	1
12	1.97	82	16	2

B.ii. Variance Decomposition of u

Forecast Horizon	*Forecast Standard Error*	*Variance Decomposition (Percentage Points)* π	u	R
1	0.23	1	99	0
4	0.64	0	98	2
8	0.79	7	82	11
12	0.92	16	66	18

B.iii. Variance Decomposition of R

Forecast Horizon	*Forecast Standard Error*	*Variance Decomposition (Percentage Points)* π	u	R
1	0.85	2	19	79
4	1.84	9	50	41
8	2.44	12	60	28
12	2.63	16	59	25

2 Application to "monetary policy measurement"

The general framework outlined above has been extensively applied to the analysis of the monetary policy transmission mechanism (by Christiano-Eichenbaum-Evans 2000, Bernanke-Mihov 1998, Bagliano-Favero 1998 among others). To highlight the key issue of identification of monetary policy actions whose effects on the economy can be analysed by means of VAR models, we introduce more details in the general formulation of a VAR system outlined above, starting from the structural and reduced forms of the model.

The need for a careful handling of the identification problem stems from the main feature of monetary policy conduct: in setting their policy instruments, monetary authorities react to current and foreseen developments in the economy. In order to capture the dynamic effects of a monetary policy action on the economy (the "monetary policy transmission mechanism"), the endogenous response of monetary policy to macroeconomic and financial variables (the "*systematic*" policy component) must be separated from the changes in the policy instruments due to "deviations" from such systematic policy (the monetary policy "*shock*"). Only when the latter is identified, the VAR analysis can yield reliable information on the transmission of policy impulses to the economy. Possible rationalizations of such shocks include changes in the "preferences" of the monetary authorities (e.g. as to the relative weight of inflation control and output stabilization goals in the loss function) and the presence of measurement errors in the data used in real time by policymakers, whereas the model is estimated on revised, error-free data.

Expanding on the above general framework, we now describe the main elements of the VAR analysis of the monetary transmission mechanism:

- the **structural form** of the model, defining the *policy rule* (systematic, endogenous response of monetary policy to developments in the economy) and the *exogenous* monetary policy shocks is written as:

$$\mathbf{A}\begin{pmatrix}\mathbf{Y}_t\\ \mathbf{P}_t\end{pmatrix}=\mathbf{C}(L)\begin{pmatrix}\mathbf{Y}_{t-1}\\ \mathbf{P}_{t-1}\end{pmatrix}+\mathbf{B}\begin{pmatrix}\mathbf{v}_t^Y\\ \mathbf{v}_t^P\end{pmatrix} \tag{1}$$

  where $\mathbf{Y}$ is a vector of non-policy macroeconomic variables (e.g. output and prices), $\mathbf{P}$ is a vector of variables directly affected by the monetary policymaker and/or containing information on the current stance of policy (e.g. interest rates and monetary aggregates), $\mathbf{v}$ is a vector of structural disturbances to the non-policy and policy variables. Matrix

A describes the contemporaneous relations among the variables, and the possibly non-zero off-diagonal elements of matrix **B** allow some structural shocks to affect directly more than one endogenous variable in the system.

- the **reduced form** of the model (VAR) to be estimated is derived as:

$$\begin{pmatrix} \mathbf{Y}_t \\ \mathbf{P}_t \end{pmatrix} = \mathbf{C}^*(L) \begin{pmatrix} \mathbf{Y}_{t-1} \\ \mathbf{P}_{t-1} \end{pmatrix} + \begin{pmatrix} \mathbf{u}_t^Y \\ \mathbf{u}_t^P \end{pmatrix} \tag{2}$$

  where $\mathbf{C}^*(L) = \mathbf{A}^{-1}\mathbf{C}(L)$ and $\mathbf{u}$ is the vector of VAR residuals (innovations in the endogenous variables), with variance-covariance matrix $E(\mathbf{u}_t\mathbf{u}_t') = \mathbf{\Sigma}$;

- the relation between the (estimated) VAR **innovations** in $\mathbf{u}$ and the (unobservable) **structural disturbances** in $\mathbf{v}$ is given by:

$$\mathbf{A}\ \mathbf{u}_t = \mathbf{B}\ \mathbf{v}_t$$

  or, equivalently

$$\mathbf{u}_t = \mathbf{A}^{-1}\mathbf{B}\ \mathbf{v}_t \tag{3}$$

  from which it is clear that the VAR residuals are linear combinations of the structural disturbances and cannot be given immediately an interpretation as fundamental economic shocks. From (3) we derive the following relationship:

$$E(\mathbf{u}_t\mathbf{u}_t') = \mathbf{A}^{-1}\mathbf{B}\, E(\mathbf{v}_t\mathbf{v}_t')\, \mathbf{B}'(\mathbf{A}^{-1})'$$

The problem of the identification of structural parameters in the system and of the structural shocks (including the *monetary policy shock*) is addressed by imposing some restrictions on the elements of **A** and **B**. In general, a model is identified by:

- assuming orthogonality of the structural disturbances in $\mathbf{v}$;

- assuming that macroeconomic variables in **Y** do not simultaneously react to policy variables in **P**, while the simultaneous feedback from **Y** to **P** is allowed;

- imposing restrictions on the contemporaneous relationships among variables in the policy block of the model reflecting the operational procedures implemented by monetary policymakers;

- imposing restrictions on the long-run response of some variables to structural disturbances.

Various combinations of these identification assumptions have been proposed in the literature. In what follows we will examine the "recursiveness" approach of Christiano, Eichenbaum and Evans (CEE, 2000), the "semi-structural" approach of Bernanke and Mihov (BM, 1998) and the long-run restrictions approach of Blanchard and Quah (BQ, 1989). Finally, a note will be devoted to an alternative identification strategy, which exploits information not contained in the endogenous variables included in the VAR system but extracted from financial markets data, usually at a high (daily) frequency (Bagliano-Favero, EER 1999; Faust-Swanson-Wright, JME 2004).