Let us start with the classical McCall model of search. This model is not only elegant, but has also become a workhorse for many questions in macro, labor and industrial organization. An important feature of the model is that it is much more tractable than the original Stigler formulation of search, as one of sampling multiple offers, but we will return to this theme below.

CHAPTER 10

The Partial Equilibrium Model

1. Basic Model

Imagine a partial equilibrium setup with a risk neutral individual in discrete time. At time $t = 0$, this individual has preferences given by

$$\sum_{t=0}^{\infty} \beta^t c_t$$

where c_t is his consumption. He starts life as unemployed. When unemployed, he has access to consumption equal to b (from home production, value of leisure or unemployment benefit). At each time period, he samples a job. All jobs are identical except for their wages, and wages are given by an exogenous stationary distribution of $F(w)$ with finite (bounded) support $\mathbb{W}$, i.e., F is defined only for $w \in \mathbb{W}$. Without loss of any generality, we can take the lower support of $\mathbb{W}$ to be 0, since negative wages can be ruled out. In other words, at every date, the individual samples a wage $w_t \in W$, and has to decide whether to take this or continue searching. Draws from $\mathbb{W}$ over time are independent and identically distributed.

This type of sequential search model can also be referred to as a model of *undirected search*, in the sense that the individual has no ability to seek or direct his search towards different parts of the wage distribution (or towards different types of jobs). This will contrast with models of *directed search* which we will see later.

Let us assume for now that there is no recall, so that the only thing the individual can do is to take the job offered within that date (with recall, the individual would be able to accumulate offers, so at time t, he can choose any of the offers he has received up at that point). If he accepts a job, he will be employed at that job forever, so the net present value of accepting a job of wage w_t is

$$\frac{w_t}{1-\beta}.$$

This is a simple decision problem. Let us specify the class of decision rules of the agent. In particular, let

$$a_t : \mathbb{W} \to [0,1]$$

denote the action of the agent at time t, which specifies his acceptance probability for each wage in $\mathbb{W}$ at time t. Let $a'_t \in \{0,1\}$ be the realization of the action by the individual (thus allowing for mixed strategies). Let also A_t denote the set of realized actions by the individual, and define $A^t = \prod_{s=0}^{t} A_s$. Then a strategy for the individual in this game is

$$p_t : A^{t-1} \times \mathbb{W} \to [0,1]$$

Let $\mathcal{P}$ be the set of such functions (with the property that $p_t(\cdot)$ is defined only if $p_s(\cdot) = 0$ for all $s \leq t$) and $\mathcal{P}^\infty$ the set of infinite sequences of such functions. The most general way of expressing the problem of the individual would be as follows. Let $\mathbb{E}$ be the expectations operator. Then the individual's problem is

$$\max_{\{p_t\}_{t=0}^{\infty} \in \mathcal{P}^\infty} \mathbb{E} \sum_{t=0}^{\infty} \beta^t c_t$$

subject to $c_t = b$ if $t < s$ and $c_t = w_s$ if $t \geq s$ where $s = \inf\{n \in \mathbb{N} : a'_n = 1\}$. Naturally, written in this way, the problem looks complicated. Nevertheless, the dynamic programming formulation of this problem will be quite tractable.

To develop this approach, let us analyze this problem by writing it recursively using dynamic programming techniques. First, let us define the value of the agent when he has sampled a job of $w \in \mathbb{W}$. This is clearly given by

$$v(w) = \max\left\{\frac{w}{1-\beta}, \beta v + b\right\}, \tag{10.1}$$

where

$$v = \int_{\mathbb{W}} v(\omega)\, dF(\omega) \tag{10.2}$$

is the continuation value of not accepting a job. Here we have made no assumptions about the structure of the set $\mathbb{W}$, which could be an interval, or might have a mass point, and the density of the distribution F may not exist. Therefore, the integral in (10.2) should be interpreted as a Lebesgue integral.

Equation (10.1) follows from the observation that the individual will either accept the job, receiving a constant consumption stream of w (valued at $w/(1-\beta)$) or will turn down this job, in which case he will enjoy the consumption level b, and receive the continuation value v. Maximization implies that the individual takes whichever of these two options gives higher net present value.

Equation (10.2), on the other hand, follows from the fact that from tomorrow on, the individual faces the same distribution of job offers, so v is simply the expected value of $v(w)$ over the stationary distribution of wages.

We are interested in finding both the value function $v(w)$ and the optimal policy of the individual.

Combining these two equations, we can write

$$v(w) = \max\left\{\frac{w}{1-\beta}, b + \beta \int_{\mathbb{W}} v(\omega)\, dF(\omega)\right\}. \tag{10.3}$$

We can now deduce the existence of optimal policies using standard theorems from dynamic programming. But in fact, (10.3) is simple enough that, one can derive these results without appealing to these theorems. In particular, this equation makes it clear that $v(w)$ must be piecewise linear with first a flat portion and then an increasing portion.

The next task is to determine the optimal policy. But the fact that $v(w)$ is non-decreasing and is piecewise linear with first a flat portion, immediately tells us that the optimal policy will take a *reservation wage* form, which is a key result of the sequential search model. More explicitly, there will exist some reservation wage R such that all wages above R will be accepted and those $w < R$ will be turned down. Moreover, this reservation wage has to be such that

$$\frac{R}{1-\beta} = b + \beta \int_{\mathbb{W}} v(\omega)\, dF(\omega), \tag{10.4}$$

so that the individual is just indifferent between taking $w = R$ and waiting for one more period. Next we also have that since $w < R$ are turned down, for all $w < R$

$$\begin{aligned} v(w) &= b + \beta \int_{\mathbb{W}} v(\omega)\, dF(\omega) \\ &= \frac{R}{1-\beta}, \end{aligned}$$

and for all $w \geq R$,

$$v(w) = \frac{w}{1-\beta}$$

Therefore,

$$\int_{\mathbb{W}} v(\omega)\, dF(\omega) = \frac{RF(R)}{1-\beta} + \int_{w \geq R} \frac{w}{1-\beta} dF(w).$$

Combining this with (10.4), we have

$$\frac{R}{1-\beta} = b + \beta \left[\frac{RF(R)}{1-\beta} + \int_{w \geq R} \frac{w}{1-\beta} dF(w) \right]$$

Manipulating this equation, we can write

$$R = \frac{1}{1-\beta F(R)} \left[b(1-\beta) + \beta \int_{R}^{+\infty} w dF(w) \right],$$

which is one way of expressing the reservation wage. More useful is to rewrite this equation as

$$\int_{w<R} \frac{R}{1-\beta} dF(w) + \int_{w \geq R} \frac{R}{1-\beta} dF(w) = b + \beta \left[\int_{w<R} \frac{R}{1-\beta} dF(w) + \int_{w \geq R} \frac{w}{1-\beta} dF(w) \right]$$

Now subtracting $\beta R \int_{w \geq R} dF(w) / (1-\beta) + \beta R \int_{w<R} dF(w) / (1-\beta)$ from both sides, we obtain

$$\begin{aligned} & \int_{w<R} \frac{R}{1-\beta} dF(w) + \int_{w \geq R} \frac{R}{1-\beta} dF(w) \\ & -\beta \int_{w \geq R} \frac{R}{1-\beta} dF(w) - \beta \int_{w<R} \frac{R}{1-\beta} dF(w) \\ = \; & b + \beta \left[\int_{w \geq R} \frac{w-R}{1-\beta} dF(w) \right] \end{aligned}$$

Collecting terms, we obtain

(10.5) $$R - b = \frac{\beta}{1-\beta} \left[\int_{w \geq R} (w-R)\, dF(w) \right],$$

which is a particularly useful and economically intuitive way of characterizing the reservation wage. The left-hand side is best understood as the cost of foregoing the wage of R, while the right hand side is the expected benefit of one more search. Clearly, at the reservation wage, these two are equal.

One implication of the reservation wage policy is that the assumption of no recall, made above, was of no consequence. In a stationary environment, the worker will have a constant reservation wage, and therefore has no desire to go back and take a job that he had previously rejected.

Let us define the right hand side of equation (10.5) as

$$g(R) \equiv \frac{\beta}{1-\beta}\left[\int_{w \geq R} (w-R)\, dF(w)\right],$$

which represents the expected benefit of one more search as a function of the reservation wage. Clearly,

$$\begin{aligned} g'(R) &= -\frac{\beta}{1-\beta}(R-R) f(R) - \frac{\beta}{1-\beta}\left[\int_{w \geq R} dF(w)\right] \\ &= -\frac{\beta}{1-\beta}[1-F(R)] < 0 \end{aligned}$$

This implies that equation (10.5) has a unique solution. Moreover, by the implicit function theorem,

$$\frac{dR}{db} = \frac{1}{1-g'(R)} > 0,$$

so that as expected, higher benefits when unemployed increase the reservation wage, making workers more picky.

Moreover, for future reference, also note that when the density of $F(R)$, denoted by $f(R)$, exists, the second derivative of g also exists and is

$$g''(R) = \frac{\beta}{1-\beta} f(R) \geq 0,$$

so that the right hand side of equation (10.5) is also convex.

The next question is to investigate how changes in the distribution of wages F affect the reservation wage. Before doing this, however, we will use this partial equilibrium McCall model to derive a very simple theory of unemployment.

2. Unemployment with Sequential Search

Let us now use the McCall model to construct a simple model of unemployment. In particular, let us suppose that there is now a continuum 1 of identical individuals sampling jobs from the same stationary distribution F. Moreover, once a job is created, it lasts until the worker dies, which happens with probability s. There is a mass of s workers born every period, so that population is constant, and these workers start out as unemployed. The death probability means that the effective discount factor of workers is equal to $\beta(1-s)$. Consequently, the value of having accepted a wage of w is:

$$v^a(w) = \frac{w}{1-\beta(1-s)}.$$

Moreover, with the same reasoning as before, the value of having a job offer at wage w at hand is

$$v(w) = \max\{v^a(w), b + \beta(1-s)v\}$$

with

$$v = \int_{\mathbb{W}} v(w)\,dF.$$

Therefore, the same steps lead to the reservation wage equation:

$$R - b = \frac{\beta(1-s)}{1-\beta(1-s)}\left[\int_{w \geq R}(w-R)\,dF(w)\right].$$

Now what is interesting is to look at the law of motion of unemployment. Let us start time t with U_t unemployed workers. There will be s new workers born into the unemployment pool. Out of the U_t unemployed workers, those who survive and do not find a job will remain unemployed. Therefore

$$U_{t+1} = s + (1-s)F(R)U_t,$$

where $F(R)$ is the probability of not finding a job (i.e., a wage offer below the reservation wage), so $(1-s)F(R)$ is the joint probability of not finding a job and surviving, i.e., of remaining unemployed. This is a simple first-order linear difference equation (only depending on the reservation wage R, which is itself independent of the level of unemployment, U_t) and determines the law of motion of unemployment. Moreover, since $(1-s)\,F(R)<1$, it is asymptotically stable, and will converge to a unique steady-state level of unemployment.

To get more insight, subtract U_t from both sides, and rearrange to obtain

$$U_{t+1}-U_t=s\,(1-U_t)-(1-s)\,(1-F(R))\,U_t.$$

This is the simplest example of the *flow approach* to the labor market, where unemployment dynamics are determined by flows in and out of unemployment. In fact is equation has the canonical form for change in unemployment in the flow approach. The left hand-side is the change unemployment (which can be either indiscreet or continuous time), while the right hand-side consists of the job destruction rate (in this case s) multiplied by $(1-U_t)$ minus the rate at which workers leave unemployment (in this case $(1-s)\,(1-F(R))$) multiplied with U_t.

The unique steady-state unemployment rate where $U_{t+1}=U_t$ is given by

$$U=\frac{s}{s+(1-s)\,(1-F(R))}.$$

This is again the canonical formula of the flow approach. The steady-state unemployment rate is equal to the job destruction rate (here the rate at which workers die, s) divided by the job destruction rate plus the job creation rate (here in fact the rate at which workers leave unemployment, which is different from the job creation rate). Clearly, an increases in s will raise steady-state unemployment. Moreover, an increase in R, that is, a higher reservation wage, will also depress job creation and increase unemployment.

3. Aside on Riskiness and Mean Preserving Spreads

To investigate the effect of changes in the distribution of wages on the reservation wage, let us introduce the concept of *mean preserving spreads*. Loosely speaking, a mean preserving spread is a change in distribution that increases risk. Let a family of distributions over some set $X\subset\mathbb{R}$ with generic element x be denoted by $F(x,r)$, where r is a shift variable, which changes the distribution function. An example will be $F(x,r)$ to stand for mean zero normal variables, with r parameterizing the variance of the distribution. In fact, the normal distribution is special in the sense that, the mean and the variance completely describe the distribution, so the notion of risk can be captured by the variance. This is generally not true. The notion of "riskier" is a more stringent notion than having a greater variance. In fact, we will see that "riskier than" is a partial order (while, clearly, comparing variances is a complete order).

Here is a natural definition of one distribution being riskier than another, first introduced by Blackwell, and then by Rothschild and Stiglitz.

Definition 10.1. *$F(x,r)$ is less risky than $F(x,r')$, written as $F(x,r) \succeq_R F(x,r')$, if for all concave and increasing $u:\mathbb{R}\to\mathbb{R}$, we have*

$$\int_X u(x)\,dF(x,r) \geq \int_X u(x)\,dF(x,r').$$

At some level, it may be a more intuitive definition of "riskiness" to require that $F(x,r)$ and $F(x,r')$ to have the same mean, i.e., $\int_X x dF(x,r) = \int_X x dF(x,r')$, while still $F(x,r) \succeq_R F(x,r')$. However, whether we do this or not is not important for our focus.

A related definition is that of second-order stochastic dominance.

Definition 10.2. *$F(x,r)$ second order stochastically dominates $F(x,r')$, written as $F(x,r) \succeq_{SD} F(x,r')$, if*

$$\int_{-\infty}^{c} F(x,r)\,dx \leq \int_{-\infty}^{c} F(x,r')\,dx, \text{ for all } c \in X.$$

In other words, this definition requires the distribution function of $F(x,r)$ to start lower and always keep a lower integral than that of $F(x,r')$. One easy case where this will be satisfied is when both distribution functions have the same mean and they intersect only once: "single crossing") with $F(x,r)$ cutting $F(x,r')$ from below.

The definitions above use weak inequalities. Alternatively, they can be strengthened to strict inequalities. In particular, the first definition would require a strict inequality for functions that are strictly concave over some range, while the second definition will require strict inequality for some c.

Theorem 10.1. ***(Blackwell, Rothschild and Stiglitz)*** *$F(x,r) \succeq_R F(x,r')$ if and only if $F(x,r) \succeq_{SD} F(x,r')$.*

Therefore, there is an intimate link between second-order stochastic dominance and the notion of riskiness. This also shows that variance is not a good measure of riskiness, since second order stochastic dominance is a partial order.

Now **mean preserving spreads** are essentially equivalent to second-order stochastic dominance with the additional restriction that both distributions have the same mean. As the term suggests, a mean preserving spread is equivalent to taking a given distribution and shifting some of the weight from around the mean to the tails. Alternative representations also include one distribution being obtained from the other by adding "white noise" to the other.

Second-order stochastic dominance plays a very important role in the theory of learning, and also more generally in the theory of decision-making under uncertainty. Here it will be useful for comparative statics.

4. Back to the Basic Partial Equilibrium Search Model

Let us return to the McCall search model. To investigate the effect of changes in the riskiness (or dispersion) of the wage distribution on reservation wages, and thus on search and unemployment behavior, let us express the reservation wage somewhat differently. Start with equation (10.5) above, which is reproduced here for convenience,

$$R - b = \frac{\beta}{1-\beta}\left[\int_{w \geq R} (w - R)\, dF(w)\right].$$

Rewrite this as

$$\begin{aligned} R-b &= \frac{\beta}{1-\beta}\left[\int_{w\geq R}(w-R)\,dF(w)\right]+\frac{\beta}{1-\beta}\left[\int_{w\leq R}(w-R)\,dF(w)\right] \\ &\quad -\frac{\beta}{1-\beta}\left[\int_{w\leq R}(w-R)\,dF(w)\right], \\ &= \frac{\beta}{1-\beta}(Ew-R)-\frac{\beta}{1-\beta}\left[\int_{w\leq R}(w-R)\,dF(w)\right], \end{aligned}$$

where Ew is the mean of the wage distribution, i.e.,

$$Ew=\int_{\mathbb{W}} wdF(w).$$

Now rearranging this last equation, we have

$$R-b=\beta(Ew-b)-\beta\int_{w\leq R}(w-R)\,dF(w).$$

Applying integration by parts to the integral on the right hand side, in particular, noting that

$$\begin{aligned} \int_{w\leq R} wdF(w) &= \int_0^R wdF(w) \\ &= wF(w)\big|_0^R-\int_0^R F(w)\,dw \\ &= RF(R)-\int_0^R F(w)\,dw, \end{aligned}$$

this equation can be rewritten as

$$R-b=\beta(Ew-b)+\beta\int_0^R F(w)\,dw. \tag{10.6}$$

Now consider a shift from F to $\tilde{F}$ corresponding to a mean preserving spread. This implies that Ew is unchanged, but by definition of a mean preserving spread (second-order stochastic dominance), the last integral increases. Therefore, the mean preserving spread induces a shift in the reservation wage from R to $\tilde{R}>R$. This reflects the greater option value of waiting when faced with a more dispersed wage distribution; lower wages are already turned down, while higher wages are now more likely.

A different way of viewing this result is that the analysis above established that the value function $v(w)$ is convex. While Theorem 10.1 shows that concave utility functions like less risky distributions, convex functions like more risky distributions.

5. Paradoxes of Search

The search framework is attractive especially when we want to think of a world without a Walrasian auctioneer, or alternatively a world with "frictions". How do prices get determined? How do potential buyers and sellers get together? Can we think of Walrasian equilibrium as an approximation to such a world under some conditions?

Search theory holds the promise of potentially answering these questions, and providing us with a framework for analysis.

5.1. The Rothschild Critique. The McCall model is an attractive starting point. It captures the intuition that individuals may be searching for the right types of job (e.g., jobs offering higher wages), trading off the prospects of future benefits (high wages) for the costs of foregoing current wages.

But everything hinges on the distribution of wages, $F(w)$. Where does this come from? Presumably somebody is offering every wage in the support of this distribution.

The basis of the Rothschild critique is that it is difficult to rationalize the distribution function $F(w)$ as resulting from profit-maximizing choices of firms.

Imagine that the economy consists of a mass 1 of identical workers similar to our searching agent. On the other side, there are N firms that can productively employ workers. Imagine that firm j has access to a technology such that it can employ l_j workers to produce

$$y_j = x_j l_j$$

units of output (with its price normalized to one as the numeraire, so that w is the real wage). Suppose that each firm can only attract workers by posting a single vacancy. Moreover, to simplify life, suppose that firms post a vacancy at the beginning of the game at $t = 0$, and then do not change the wage from then on. This will both simplify the strategies, and imply that the wage distribution will be stationary, since all the same wages will remain active throughout time. [Can you see why this simplifies the discussion? Imagine, for contrast, the case in which each firm only hires one worker; then think of the wage distribution at time t, $F_t(w)$, starting with some arbitrary $F_0(w)$. Will it remain constant?]

Suppose that the distribution of x in the population of firms is given by $G(x)$ with support $X \subset \mathbb{R}_+$. Also assume that there is some cost $\gamma > 0$ of posting a vacancy at the beginning, and finally, that $N >> 1$ (i.e., $N = \int_{-\infty}^{\infty} dG(x) >> 1$) and each worker samples one firm from the distribution of posting firms.

As before, we will assume that once a worker accepts a job, this is permanent, and he will be employed at this job forever. Moreover let us set $b = 0$, so that there is no unemployment benefits. Finally, to keep the environment entirely stationary, assume that once a worker accepts a job, a new worker is born, and starts search.

Will these firms offer a non-degenerate wage distribution $F(w)$?

The answer is no.

First, note that an endogenous wage distribution equilibrium would correspond to a function

$$p : X \rightarrow \{0, 1\},$$

denoting whether the firm is posting a vacancy or not, and if it is, i.e., $p = 1$,

$$h : X \rightarrow \mathbb{R}_+,$$

specifying the wage it is offering.

It is intuitive that $h(x)$ should be non-decreasing (higher wages are more attractive to high productivity firms). Let us suppose that this is so, and denote its set-valued inverse mapping by h^{-1}. Then, the along-the-equilibrium path wage distribution is

$$F(w) = \frac{\int_{-\infty}^{h^{-1}(w)} p(x)\, dG(x)}{\int_{-\infty}^{\infty} p(x)\, dG(x)}.$$

Why?

In addition, the strategies of workers can be represented by a function

$$a : \mathbb{R}_+ \rightarrow [0, 1]$$

denoting the probability that the worker will accept any wage in the "potential support" of the wage distribution, with 1 standing for acceptance. This is general enough to nest non-symmetric or mixed strategies.

The natural equilibrium concept is subgame perfect Nash equilibrium, whereby the strategies of firms (p, h) and those of workers, a, are best responses to each other in all subgames.

The same arguments as above imply that all workers will use a reservation wage, so

$$\begin{aligned} a(w) &= 1 \text{ if } w \geq R \\ &= 0 \text{ otherwise} \end{aligned}$$

Since all workers are identical and the equation above determining the reservation wage, (10.5), has a unique solution, all workers will all be using the same reservation rule, accepting all wages $w \geq R$ and turning down those $w < R$. Workers' strategies are therefore again characterized by a reservation wage R.

Now take a firm with productivity x offering a wage $w' > R$. Its net present value of profits from this period's matches is

$$\pi(p = 1, w' > R, x) = -\gamma + \frac{1}{n} \frac{(x - w')}{1 - \beta}$$

where

$$n = \int_{-\infty}^{\infty} p(x) \, dG(x)$$

is the measure of active firms, $1/n$ is the probability of a match within each period (since the population of active firms and searching workers are constant), and $x - w'$ is the profit from the worker discounted at the discount factor β.

Notice two (implicit) assumptions here: (1) wage posting: each job comes with a commitment to a certain wage; (2) undirected search: the worker makes a random draw from the distribution F, and the only way he can seek higher wages is by turning down lower wages that he samples.

This firm can deviate and cut its wage to some value in the interval $[R, w')$. All workers will still accept this job since its wage is above the reservation wage, and the firm will increase its profits to

$$\pi\left(p=1, w \in [R, w'), x\right) = -\gamma + \frac{1}{n}\frac{x-w}{1-\beta} > \pi\left(p=1, w', x\right)$$

So there should not be any wages strictly above R.

Next consider a firm offering a wage $\tilde{w} < R$. This wage will be rejected by all workers, and the firm would lose the cost of posting a vacancy, i.e.,

$$\pi\left(p=1, w<R, x\right) = -\gamma,$$

and this firm can deviate to $p = 0$ and make zero profits. Therefore, in equilibrium when workers use the reservation wage rule of accepting only wages greater than R, all firms will offer the same wage R, and there is no distribution and no search.

This establishes

THEOREM 10.2. *When all workers are homogeneous and engage in undirected search, all equilibrium distributions will have a mass point at their reservation wage R.*

In fact, the paradox is even deeper.

5.2. The Diamond Paradox. The following result is one form of the Diamond paradox:

THEOREM 10.3. ***(Diamond Paradox)*** *For all $\beta < 1$, the unique equilibrium in the above economy is $R = 0$.*

Given the Theorem 10.2, this result is easy to understand. Theorem 10.2 implies that all firms will offer the same wage, R.

Suppose $R > 0$, and $\beta < 1$. What is the optimal acceptance function, a, for a worker?

If the answer is

$$
\begin{aligned}
a(w) &= 1 \text{ if } w \geq R \\
&= 0 \text{ otherwise}
\end{aligned}
$$

then we can support all firms offering $w = R$ as an equilibrium (notice that the acceptance function needs to be defined for wages "off-the-equilibrium path"). Why is this important?

However, we can prove:

LEMMA 10.1. *There exists $\varepsilon > 0$ such that when "almost all" firms are offering $w = R$, it is optimal for each worker to use the following acceptance strategy:*

$$
\begin{aligned}
a(w) &= 1 \textit{ if } w \geq R - \varepsilon \\
&= 0 \textit{ otherwise}
\end{aligned}
$$

Note: think about what "almost all" means here and why it is necessary.

PROOF. If the worker accepts the wage of $R - \varepsilon$ today his payoff is

$$u^{accept} = \frac{R - \varepsilon}{1 - \beta}$$

If he rejects and waits until next period, then since "almost all" firms are offering R, he will receive the wage of R, so

$$u^{reject} = \frac{\beta R}{1 - \beta}$$

where the additional β comes in because of the waiting period. For all $\beta < 1$, there exists $\varepsilon > 0$ such that

$$u^{accept} > u^{reject},$$

proving the claim. □

What is the intuition for this lemma?

But this implies that, starting from an allocation where all firms offer R, any firm can deviate and offer a wage of $R - \varepsilon$ and increase its profits. This proves that no wage $R > 0$ can be the equilibrium, proving the proposition.

Notice that subgame perfection is important here. We know that these are non-subgame perfect Nash equilibria, and this highlights the importance of using the right equilibrium concept in the context of dynamic economies.

So now we are in a conundrum. Not only does there fail to be a wage distribution, but irrespective of the distribution of productivities or the degree of discounting, all firms offer the lowest possible wage, i.e., they are full monopsonists.

How do we resolve this paradox?

(1) By assumption: assume that $F(w)$ is not the distribution of wages, but the distribution of "fruits" exogenously offered by "trees". This is clearly unsatisfactory, both from the modeling point of view, and from the point of view of asking policy questions from the model (e.g., how does unemployment insurance affect the equilibrium? The answer will depend also on how the equilibrium wage distribution changes).

(2) Introduce other dimensions of heterogeneity: to be done later.

(3) Modify the wage determination assumptions: to be done in a little bit.

CHAPTER 11

Basic Equilibrium Search Framework

1. Motivation

Importance of labor market flows, job creation, job destruction.

Need for a framework that can be used for equilibrium analysis, but allows for unemployment→Equilibrium search models.

More reduced form than a partial equilibrium model in order to avoid the "paradoxes" mentioned above.

2. The Basic Search Model

Now we discuss the basic search-matching model, or sometimes called the flow approach to the labor market.

Here the basic idea is that that are frictions in the labor market, making it costly (time-consuming) for workers to find firms and vice versa. This will lead to what is commonly referred to as "frictional unemployment". However, as soon as there are these types of frictions, there are also quasi-rents in the relationship between firms and workers, and there will be room for rent-sharing. In the basic search model, the main reason for high unemployment may not be the time costs of finding partners, but bargaining between firms and workers which leads to non-market-clearing equilibrium prices.

Here is a simple version of the basic search model.

The first important object is the matching function, which gives the number of matches between firms and workers as a function of the number of unemployed workers and number of vacancies.

Matching Function: $\quad$ Matches $= x(U, V)$

This function captures the frictions inherent in the process of assigning workers to jobs in a very *reduced form* way. This reduced-form structure is its advantage and disadvantage. It is difficult to have microfoundations for this function, but it is very tractable, fairly easy to map to data (at least to data on job flows and worker flows), and captures the intuitive notion that job finding rates for workers should depend on how many unemployed workers are chasing how many vacancies.

Of course the form of the matching function will also depend on what the time horizon is.

Following our treatment of the Shapiro-Stiglitz model, we will work with continuous time, so we should think of $x(U,V)$ as the flow rate of matches.

We typically assume that this matching function exhibits constant returns to scale (CRS), that is,

$$\begin{aligned} \text{Matches} &= xL = x(uL, vL) \\ &\implies x = x(u,v) \end{aligned}$$

Here we have adopted the usual notation:

U =unemployment;

u =unemployment rate

V =vacancies;

v = vacancy rate (per worker in labor force)

L = labor force

Existing aggregate evidence suggests that the assumption of x exhibiting CRS is reasonable (Blanchard and Diamond, 1989)

Using the constant returns assumption, we can express everything as a function of the tightness of the labor market.

Therefore;

$$q(\theta) \equiv \frac{x}{v} = x\left(\frac{u}{v}, 1\right),$$

where $\theta \equiv v/u$ is the tightness of the labor market

Since we are in continuous time, these things immediately map to flow rates. Namely

$$q(\theta) : \text{Poisson arrival rate of match for a vacancy}$$
$$q(\theta)\theta : \text{Poisson arrival rate of match for an unemployed worker}$$

What does Poisson mean?

Take a short period of time Δt, then the Poisson process is defined such that during this time interval, the probability that there will be one arrival, for example one arrival of a job for a worker, is

$$\Delta t q(\theta)\theta$$

The probability that there will be more than one arrivals is vanishingly small (formally, of order $o(\Delta t)$).

Therefore,

$1 - \Delta t\theta q(\theta)$: probability that a worker looking for a job will not find one during Δt

This probability depends on θ, thus leading to a potential externality—the search behavior of others affects my own job finding rate.

The search model is also sometimes called the flow approach to unemployment because it's all about job flows. That is about job creation and job destruction.

This is another dividing line between labor and macro. Many macroeconomists look at data on job creation and job destruction following Davis and Haltiwanger. Most labor economists do not look at these data. Presumably there is some information in them.

Job creation is equal to

$$\text{Job creation} = u\theta q(\theta)L$$

What about job destruction?

Let us start with the simplest model of job destruction, which is basically to treat it as "exogenous".

Think of it as follows, firms are hit by adverse shocks, and then they decide whether to destroy or to continue.

$\longrightarrow$ Adverse Shock$\longrightarrow$destroy

$\longrightarrow$ continue

Exogenous job destruction: Adverse shock $= -\infty$ with "probability " s

As in the Shapiro-Stiglitz model, we will focus on steady states.

Steady State:

flow into unemployment = flow out of unemployment

Therefore, with exogenous job destruction:

$$s(1-u) = \theta q(\theta) u$$

This gives the steady-state unemployment rate as

$$u = \frac{s}{s + \theta q(\theta)}$$

This relationship is sometimes referred to as the Beveridge Curve, or the U-V curve. It draws a downward sloping locus of unemployment-vacancy combinations in the U-V space that are consistent with flow into unemployment being equal with flow out of unemployment. Some authors interpret shifts of this relationship is reflecting structural changes in the labor market, but we will see that there are many factors that might actually shift at a generalized version of such relationship.

It is a crucial equation even if you don't like the search model. It relates the unemployment rate to the rate at which people leave their jobs and and unemployment and the rate at which people leave the unemployment pool.

In a more realistic model, of course, we have to take into account the rate at which people go and come back from out-of-labor force status.

Let's next turn to the production side.

Let the output of each firm be given by neoclassical production function combining labor and capital:

$$Y = AF(K, N)$$

where the production function F is assumed to exhibit constant returns, K is the capital stock of the economy, and N is employment (different from labor force because of unemployment).

Defining $k \equiv K/N$ as the capital labor ratio, we have that output per worker is:

$$\frac{Y}{N} = Af(k) \equiv AF(\frac{K}{N}, 1)$$

because of constant returns.

Two interpretations $\longrightarrow$ each firm is a "job" hires one worker
each firm can hire as many worker as it likes

For our purposes either interpretation is fine

Hiring: Vacancy costs γ_0: fixed cost of hiring

r: cost of capital

δ: depreciation

The key assumption here is that capital is perfectly reversible.

As in the Shapiro Stiglitz model, we will solve everything by using dynamic programming, or in other words by writing the asset value equations. As in there, let us define those in terms of the present discounted values.

Namely, let

J^V : PDV of a vacancy

J^F :PDV of a "job"

J^U :PDV of a searching worker

J^E :PDV of an employed worker

More generally, we have that worker utility is: $EU_0 = \int_0^\infty e^{-rt}U(c_t)$, but for what we care here, risk-neutrality is sufficient.

Utility $U(c) = c$, in other words, linear utility, so agents are risk-neutral.

Perfect capital market gives the asset value for a vacancy (in steady state) as

$$rJ^V = -\gamma_0 + q(\theta)(J^F - J^V)$$

Intuitively, there is a cost of vacancy equal to γ_0 at every instant, and the vacancy turns into a filled job at the flow rate $q(\theta)$.

Notice that in writing this expression, we have assumed that firms are risk neutral. Why is this important?

$\longrightarrow$ workers risk neutral, or

$\longrightarrow$ complete markets

The question is how to model job creation (which is the equivalent of how to model labor demand in a competitive labor market).

Presumably, firms decide to create jobs when there are profit opportunities.

The simplest and perhaps the most extreme form of endogenous job creation is to assume that there will be a firm that creates a vacancy as soon as the value of a vacancy is positive (after all, unless there are scarce factors necessary for creating vacancies anybody should be able to create one).

This is sometimes referred to as the free-entry assumption, because it amounts to imposing that whenever there are potential profits they will be eroded by entry.

Free Entry $\Longrightarrow$

$$J^V \equiv 0$$

The most important implication of this assumption is that job creation can happen really "fast", except because of the frictions created by matching searching workers to searching vacancies.

Alternative would be: $\gamma_0 = \Gamma_0(V)$ or $\Gamma_1(\theta)$, so as there are more and more jobs created, the cost of opening an additional job increases.

Free entry implies that

$$J^F = \frac{\gamma_0}{q(\theta)}$$

Next, we can write another asset value equation for the value of a field job:

$$r(J^F + k) = Af(k) - \delta k - w - s(J^F - J^V)$$

Intuitively, the firm has two assets: the fact that it is matched with a worker, and its capital, k. So its asset value is $J^F + k$ (more generally, without the perfect reversability, we would have the more general $J^F(k)$). Its return is equal to production, $Af(k)$, and its costs are depreciation of capital and wages, δk and w. Finally, at the rate s, the relationship comes to an end and the firm loses J^F.

Perfect Reversability implies that w does not depend on the firm's choice of capital

$\Longrightarrow$ equilibrium capital utilization $f'(k) = r + \delta$— Modified Golden Rule

[...Digression: Suppose k is not perfectly reversible then suppose that the worker captures a fraction β all the output in bargaining. Then the wage depends on the capital stock of the firm, as in the holdup models discussed before.

$$\begin{aligned} w(k) &= \beta Af(k) \\ Af'(k) &= \frac{r+\delta}{1-\beta} \text{ ; capital accumulation is distorted} \end{aligned}$$

...]

Now, ignoring this digression

$$Af(k) - (r-\delta)k - w - \frac{(r+s)}{q(\theta)}\gamma_0 = 0$$

Now returning to the worker side, the risk neutrality of workers gives

$$rJ^U = z + \theta q(\theta)(J^E - J^U)$$

where z is unemployment benefits. The intuition for this equation is similar. We also have

$$rJ^E = w + s(J^U - J^E)$$

Solving these equations we obtain

$$\begin{aligned} rJ^U &= \frac{(r+s)z + \theta q(\theta)w}{r+s+\theta q(\theta)} \\ rJ^E &= \frac{sz + [r + \theta q(\theta)]\, w}{r+s+\theta q(\theta)} \end{aligned}$$

How are wages determined? Nash Bargaining.

Why do we need bargaining? Answer: bilateral monopoly or much more specifically: match specific surplus.

Think of a competitive labor market, at the margin the firm is indifferent between employing the marginal worker or not, and the worker is indifferent between supplying the marginal hour or not (or working for this firm or another firm). We can make both parties in different at the same time—no match-specific surplus.

In a frictional labor market, if we choose the wage such that $J^E = 0$, we will typically have $J^F > 0$ and vice versa. There is some surplus to be shared.

Nash solution to bargaining is again the natural benchmark. Let us assume that the worker has bargaining power β.

Applying this formula, for pair i, we have

$$\begin{aligned} rJ_i^F &= Af(k) - (r+\delta)k - w_i - sJ_i^F \\ rJ_i^E &= w_i - s(J_i^E - J_0^U). \end{aligned}$$

The Nash solution will solve

$$\max (J_i^E - J^U)^\beta (J_i^F - J^V)^{1-\beta}$$

$$\beta = \text{bargaining power of the worker}$$

Since we have linear utility, thus "transferable utility", this implies

$$\Longrightarrow \quad J_i^E - J^U = \beta(J_i^F + J_i^E - J^V - J^U)$$
$$\Longrightarrow \quad w = (1-\beta)z + \beta\left[Af(k) - (r+\delta)k + \theta\gamma_0\right]$$

Here $[Af(k) - (r+\delta)k + \theta\gamma_0]$ is the quasi-rent created by a match that the firm and workers share. Why is the term $\theta\gamma_0$ there?

Now we are in this position to characterize the steady-state equilibrium.

Steady State Equilibrium is given by four equations

(1) The Beveridge curve:

$$u = \frac{s}{s + \theta q(\theta)}$$

(2) Job creation leads zero profits:

$$Af(k) - (r+\delta)k - w - \frac{(r+s)}{q(\theta)}\gamma_0 = 0$$

(3) Wage determination:

$$w = (1-\beta)z + \beta\left[Af(k) - (r+\delta)k + \theta\gamma_0\right]$$

(4) Modified golden rule:

$$Af'(k) = r + \delta$$

These four equations define a block recursive system

$(4) + r \longrightarrow k$

$k + r + (2) + (3) \longrightarrow \theta, w$

$\theta + (1) \longrightarrow u$

Alternatively, combining three of these equations we obtain the zero-profit locus, the VS curve, and combine it with the Beveridge curve. More specifically,

$(2), (3), (4) \Longrightarrow$ the VS curve

$$(1-\beta)\left[Af(k)-(r+\delta)k-z\right]-\frac{r+\delta+\beta\theta q(\theta)}{q(\theta)}\gamma_0=0$$

Therefore, the equilibrium looks very similar to the intersection of "quasi-labor demand" and "quasi-labor supply".

Quasi-labor supply is given by the Beveridge curve, while labor demand is given by the zero profit conditions.

Given this equilibrium, comparative statics (for steady states) are straightforward.

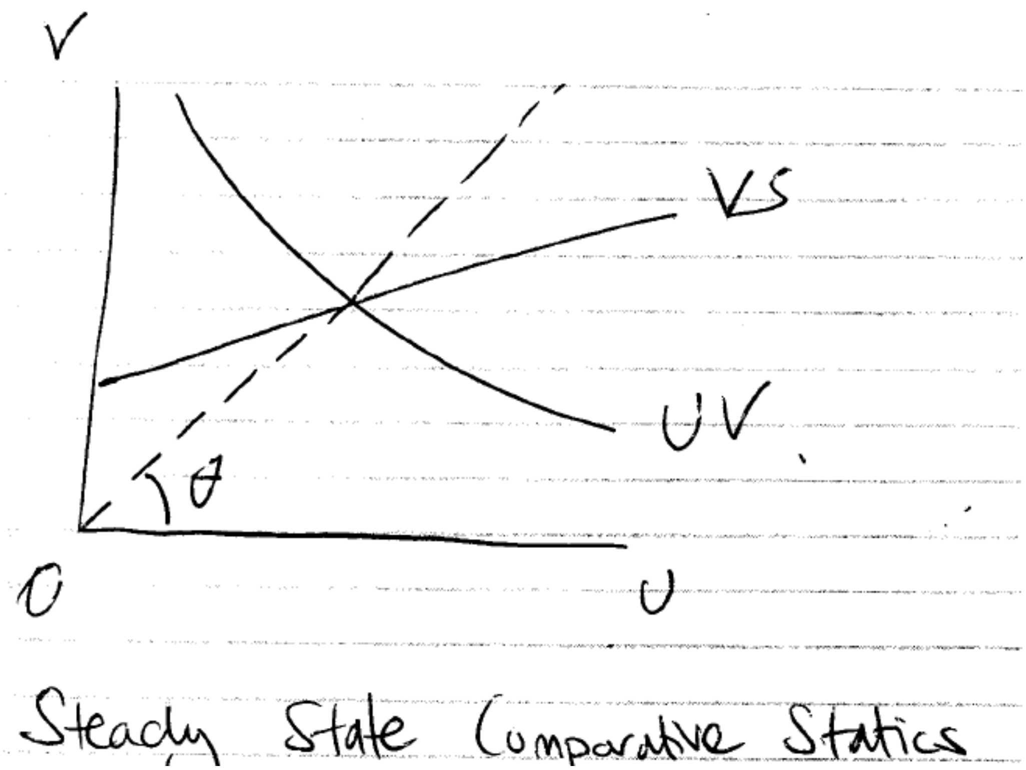

FIGURE 11.1

For example:

$s\uparrow \quad U\uparrow \quad V\uparrow \quad \theta\downarrow \quad w\downarrow$

$r\uparrow \quad U\uparrow \quad V\downarrow \quad \theta\downarrow \quad w\downarrow$

$\gamma_0\uparrow \quad U\uparrow \quad V\downarrow \quad \theta\downarrow \quad w\downarrow$

$\beta \uparrow$	$U \uparrow$	$V \downarrow$	$\theta \downarrow$	$w \uparrow$
$z \uparrow$	$U \uparrow$	$V \downarrow$	$\theta \downarrow$	$w \uparrow$
$A \uparrow$	$U \downarrow$	$V \uparrow$	$\theta \uparrow$	$w \uparrow$

Thus, a greater exogenous separation rate, higher discount rates, higher costs of creating vacancies, higher bargaining power of workers, higher unemployment benefits lead to higher unemployment. Greater productivity of jobs, leads to lower unemployment.

Interestingly, some of those, notably the greater separation rate also increases the number of vacancies.

Can we think of any of these factors is explaining the rise in unemployment in Europe during the 1980s, or the lesser rise in unemployment in 1980s in in the United States?

3. Efficiency of Search Equilibrium

Is the search equilibrium efficient? Clearly, it is inefficient relative to a first-best alternative, e.g., a social planner that can avoid the matching frictions.

However, this is not an interesting benchmark. Much more interesting is whether a social planner affected by exactly the same externalities as the market economy can do better than the decentralized equilibrium.

An alternative way of asking this question is to think about externalities. In this economy there are two externalities

$$\begin{array}{lll} \theta \uparrow & \Longrightarrow & \text{workers find jobs more easily} \\ & \hookrightarrow & \text{thick-market externality} \\ & \Longrightarrow & \text{firms find workers more slowly} \\ & \hookrightarrow & \text{congestion externality} \end{array}$$

Therefore, the question of efficiency boils down to whether these two externalities cancel each other or whether one of them dominates.

To analyze this question more systematically, consider a social planner subject to the same constraints, intending to maximize "total surplus", in other words, pursuing a utilitarian objective.

First ignore discounting, i.e., $r \to 0$, then the planner's problem can be written as

$$\begin{aligned} \max_{u,\theta} SS &= (1-u)y + uz - u\theta\gamma_0. \\ & \text{s.t.} \\ u &= \frac{s}{s+\theta q(\theta)}. \end{aligned}$$

where we assumed that z corresponds to the utility of leisure rather than unemployment benefits (how would this be different if z were unemployment benefits?)

The form of the objective function is intuitive. For every employed worker, a fraction $1-u$ of the workers, the society receives an output of y; for every unemployed worker, a fraction u of the population, it receives z, and in addition for every vacancy it pays the cost of γ_0 (and there are $u\theta$ vacancies).

The constraint on this problem is that imposed by the matching frictions, i.e. the Beveridge curve, capturing the fact that lower unemployment can only be achieved by creating more vacancies, i.e., higher θ.

Holding $r = 0$, turns this from a dynamic into a static optimization problem, and it can be analyzed by forming the Lagrangian, which is

$$\mathcal{L} = (1-u)y + uz - u\theta\gamma_0 + \lambda \left[u - \frac{s}{s+\theta q(\theta)} \right]$$

The first-order conditions with respect to u and θ are straightforward:

$$(y-z) + \theta\gamma_0 = \lambda$$

$$u\gamma_0 = \lambda s \frac{\theta q'(\theta) + q(\theta)}{(s+\theta q(\theta))^2}$$

Since the constraint will clearly binding (why is this? Otherwise reduce θ, and social surplus increases), we can substitute for u from the Beveridge curve, and obtain:

$$\lambda = \frac{\gamma_0 (s + \theta q(\theta))}{\theta q'(\theta) + q(\theta)}$$

Now substitute this into the first condition to obtain

$$[\theta q'(\theta) + q(\theta)](y-z) + [\theta q'(\theta) + q(\theta)]\theta\gamma_0 - \gamma_0 (s + \theta q(\theta)) = 0$$

Now simplifying and dividing through by $q(\theta)$, we obtain

$$[1-\eta(\theta)]\,[y-z]-\frac{s+\eta(\theta)\theta q(\theta)}{q(\theta)}\gamma_0=0.$$

where

$$\eta(\theta)=-\frac{\theta q'(\theta)}{q(\theta)}=\frac{\frac{\partial M(U,V)}{\partial U}U}{M(U,V)}$$

is the elasticity of the matching function respect to unemployment.

Recall that in equilibrium, we have (with $r=0$)

$$(1-\beta)(y-z)-\frac{s+\beta\theta q(\theta)}{q(\theta)}\gamma_0=0.$$

Comparing these two conditions we find that efficiency obtains if and only if

$$\beta=\eta(\theta).$$

In other words, efficiency requires the bargaining power of the worker to be equal to the elasticity of the matching function with respect to unemployment.

We can also note that this result is made possible by the fact that the matching function is constant returns to scale, and efficiency would never obtain if it exhibited increasing or increasing returns to scale. (Why is this? How would go about proving this?)

The condition $\beta=\eta(\theta)$ is the famous *Hosios condition*. It requires the bargaining power of a factor to be equal to the elasticity of the matching function with respect to the corresponding factor.

What is the intuition?

It is not easy to give an intuition for this result, but here is an attempt: as a planner you would like to increase the number of vacancies to the point where the marginal benefit in terms of additional matches is equal to the cost. In equilibrium, vacancies enter until the marginal benefits in terms of their bargained returns is equal to the cost. So if β is too high, they are getting too small a fraction of the return, and they will not enter enough. If β is too low, then they are getting too much of the surplus, so there will be excess entry. The right value of β turns out to be the one that is equal to the elasticity of the matching function with respect to unemployment (thus $1-\beta$ is equal to the elasticity of the matching function with respect to vacancies, by constant returns to scale).

Exactly the same result holds when we have discounting, i.e., $r > 0$

In this case, the objective function is

$$SS^* = \int_0^\infty e^{-rt} \left[Ny - zN - \gamma_0 \theta (L-N) \right] dt$$

and will be maximized subject to

$$\dot{N} = q(\theta)\theta(L-N) - sN$$

The first-order condition is

$$y - z - \frac{r + s + \eta(\theta) q(\theta)\theta}{q(\theta)\left[1-\eta(\theta)\right]} \gamma_0 = 0$$

Compared to the equilibrium where

$$(1-\beta)[y-z] + \frac{r+s+\beta q(\theta)\theta}{q(\theta)} \gamma_0 = 0$$

Again, $\eta(\theta) = \beta$ would decentralized the constrained efficient allocation.

At this point, you may be puzzled. Isn't there unemployment in equilibrium? So the equilibrium being efficient means that the social planner likes unemployment too. This raises the question: What is the use of unemployment?

The answer to this question is quite revealing. Unemployment in fact has a social role in this model. Its role is to facilitate trade at low transaction costs; the greater is unemployment, the less costly this is to fill vacancies (which are in turn costly to open). This highlights why the bargaining parameter should be related to the elasticity of the matching function. The greater is this elasticity, it means that the more important it is to have more unemployed workers around to facilitate matching, and that means a high shadow value of unemployed workers, which corresponds to a high β in equilibrium.

4. Endogenous Job Destruction

So far we treated the rate at which jobs get destroyed as a constant, s, giving us a simple equation

$$\dot{u} = s(1-u) - \theta q(\theta) u$$

But presumably thinking of job destruction as exogenous is not satisfactory. Firms decide when to expand and contract, so it's a natural next step to endogenize s.

To do this, suppose that each firm consists of a single job (so we are now taking a position on for size). Also assume that the productivity of each firm consists of two components, a common productivity and a firm-specific productivity.

In particular

$$\text{productivity for firm } i = \underbrace{p}_{\text{common productivity}} + \underbrace{\sigma \times \varepsilon_i}_{\text{firm-specific}}$$

where

$$\varepsilon_i \sim F(\cdot)$$

over support $\underline{\varepsilon}$ and $\bar{\varepsilon}$, and σ is a parameter capturing the importance of firm-specific shocks.

Moreover, suppose that each new job starts at $\varepsilon = \bar{\varepsilon}$, but does not necessarily stay there. In particular, there is a new draw from $F(\cdot)$ arriving at the flow the rate λ.

To simplify the discussion, let us ignore wage determination and set

$$w = b$$

This then gives the following value function (written in steady state) for a an active job with productivity shock ε (though this job may decide not to be active):

$$rJ^F(\varepsilon) = p + \sigma\varepsilon - b + \lambda \left[\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \max\{J^F(x), J^V\} dF(x) - J^F(\varepsilon) \right]$$

where J^V is the value of a vacant job, which is what the firm becomes if it decides to destroy. The max operator takes care of the fact that the firm has a choice after the realization of the new shock, x, whether to destroy or to continue.

Since with free entry $J^V = 0$, we have

$$rJ^F(\varepsilon) = p + \sigma\varepsilon - b + \lambda\left[E(J^F) - J^F(\varepsilon)\right] \tag{11.1}$$

where now we write $J^F(\varepsilon)$ to denote the fact that the value of employing a worker for a firm depends on firm-specific productivity.

$$E(J^F) = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \max\left\{J^F(x), 0\right\} dF(x) \tag{11.2}$$

is the expected value of a job after a draw from the distribution $F(\varepsilon)$.

Given the Markov structure, the value conditional on a draw does not depend on history.

What is the intuition for this equation?

Differentiation of (11.1) immediately gives

$$\frac{dJ^F(\varepsilon)}{d\varepsilon} = \frac{\sigma}{r+\lambda} > 0 \tag{11.3}$$

Greater productivity gives greater values the firm.

When will job destruction take place?

Since (11.3) establishes that J^F is monotonic in ε, job destruction will be characterized by a cut-off rule, i.e.,

$$\exists\, \varepsilon_d : \varepsilon < \varepsilon_d \longrightarrow \text{ destroy}$$

Clearly, this cutoff threshold will be defined by

$$rJ^F(\varepsilon_d) = 0$$

But we also have $rJ^F(\varepsilon_d) = p + \sigma\varepsilon_d - b + \lambda\left[E(J^F) - J^F(\varepsilon_d)\right]$, which yields an equation for the value of a job after a new draw:

$$E(J^F) = -\frac{p + \sigma\varepsilon_d - b}{\lambda} > 0$$

This is an interesting result; it implies that since the expected value of continuation is positive (remember equation (11.2)), the flow profits of the marginal job, $p+\sigma\varepsilon_d-b$, must be negative. Why is this? The answer is option value. Continuing as a productive unit means that the firm has the option of getting a better draw in the future, which is potentially profitable. For this reason it waits until current profits are sufficiently negative to destroy the job; in other words there is a natural form of labor hoarding in this economy.

Furthermore, we have a tractable equation for $J^F(\varepsilon)$:

$$J^F(\varepsilon) = \frac{\sigma}{r+\lambda}(\varepsilon - \varepsilon_d)$$

Let us now make more progress towards characterizing $E(J^F)$

By definition, we have

$$E(J^F) = \int_{\varepsilon_d}^{\bar{\varepsilon}} J^F(x) dF(x)$$

(where we have used the fact that when $\varepsilon < \varepsilon_d$, the job will be destroyed).

Now doing integration by parts, we have

$$\begin{aligned} E(J^F) &= \int_{\varepsilon_d}^{\bar{\varepsilon}} J^F(x) dF(x) = J^F(x)F(x)\Big|_{\varepsilon_d}^{\bar{\varepsilon}} - \int_{\varepsilon_d}^{\bar{\varepsilon}} F(x)\frac{dJ^F(x)}{dx}dx \\ &= J^F(\bar{\varepsilon}) - \frac{\sigma}{\lambda+r}\int_{\varepsilon_d}^{\bar{\varepsilon}} F(x)dx \\ &= \frac{\sigma}{\lambda+r}\int_{\varepsilon_d}^{\bar{\varepsilon}} [1-F(x)]\, dx \end{aligned}$$

where the last line use the fact that $J^F(\varepsilon) = \frac{\sigma}{\lambda+r}(\varepsilon - \varepsilon_d)$, so incorporates $J^F(\bar{\varepsilon})$ into the integral

Next, we have that

$$\begin{aligned} \underbrace{p + \sigma\varepsilon_d - b}_{\text{profit flow from marginal job}} &= -\frac{\lambda\sigma}{r+\lambda}\int_{\varepsilon_d}^{\bar{\varepsilon}} [1-F(x)]\, dx \\ &< 0 \text{ due to option value} \end{aligned}$$

which again highlights the hoarding result. More importantly, we have

$$\frac{d\varepsilon_d}{d\sigma} = \frac{p-b}{\sigma}\left[\sigma\left(\frac{r+\lambda F(\varepsilon_d)}{r+\lambda}\right)\right]^{-1} > 0.$$

which implies that when there is more dispersion of firm-specific shocks, there will be more job destruction

The job creation part of this economy is similar to before. In particular, since firms enter at the productivity $\bar{\varepsilon}$, we have

$$\begin{aligned} q(\theta) J^F(\bar{\varepsilon}) &= \gamma_0 \\ &\Longrightarrow \frac{\gamma_0(r+\lambda)}{\sigma(\bar{\varepsilon}-\varepsilon_d)} = q(\theta) \end{aligned}$$

Recall that as in the basic search model, job creation is "sluggish", in the sense that it is dictated by the matching function; it cannot jump it can only increase by investing more resources in matching.

On the other hand, job destruction is a jump variable so it has the potential to adjust much more rapidly (this feature was emphasized a lot when search models with endogenous job-destruction first came around, because at the time the general belief was that job destruction rates were more variable than job creation rates; now it's not clear whether this is true; it seems to be true in manufacturing, but not in the whole economy).

The Beveridge curve is also different now. Flow into unemployment is also endogenous, so in steady-state we need to have

$$\lambda F(\varepsilon_d)(1-u) = q(\theta)\theta u$$

In other words:

$$u = \frac{\lambda F(\varepsilon_d)}{\lambda F(\varepsilon_d) + q(\theta)\theta},$$

which is very similar to our Beveridge curve above, except that $\lambda F(\varepsilon_d)$ replaces s.

The most important implication of this is that shocks (for example to productivity) now also shift the Beveridge curve shifts. For example, an increase in p will cause an inward shift of the Beveridge curve; so at a given level of creation, unemployment will be lower.

How do you think endogenous job destruction affects efficiency?

5. A Two-Sector Search Model

Now consider a two-sector version of the search model, where there are skilled and unskilled workers. In particular, suppose that the labor force consists of L_1 and L_2 workers, i.e.

L_1 : unskilled worker

L_2 : skilled worker

Firms decide whether to open a skilled vacancy or an unskilled vacancy.

$$\left.\begin{array}{l} M_1 = x(U_1, V_1) \\ M_2 = x(U_2, V_2) \end{array}\right\} \text{ the same matching function in both sectors.}$$

Opening vacancies is costly in both markets with

γ_1 : cost of vacancy for unskilled worker

γ_2 : cost of vacancy for skilled worker.

As before, shocks arrive at some rate, here assumed to be exogenous and potentially different between the two types of jobs

$$s_1, s_2 : \text{ separation rates}$$

Finally, we allow for population growth of both skilled unskilled workers to be able to discuss changes in the composition of the labor force. In particular, let the rate of population growth of L_1 and L_2 be n_1 and n_2 respectively.

$$n_1, n_2 : \text{ population growth rates}$$

This structure immediately implies that there will be two separate Beveridge curves for unskilled and skilled workers, given by

$$u_1 = \frac{s_1 + n_1}{s_1 + n_1 + \theta_1 q(\theta_1)} \qquad u_2 = \frac{s_2 + n_2}{s_2 + n_2 + \theta_2 q(\theta_2)}.$$

(can you explain these equations? Derive them?)

So different unemployment rates are due to three observable features, separation rates, population growth and job creation rates.

The production side is largely the same as before

$$\text{output} \quad Af(K, N)$$

where N is the effective units of labor, consisting of skilled and unskilled workers.

We assumed that each unskilled worker has one unit of effective labor, while each skilled worker has $\eta > 1$ units of effective labor.

Finally, the interest rate is still r and the capital depreciation rate is δ.

Asset Value Equations are as before.

For filled jobs,

$$\begin{aligned} rJ_1^F &= Af(k) - (r+\delta)k - w_1 - s_1 J_1^F \\ rJ_2^F &= Af(k)\eta - (r+\delta)k\eta - w_2 - s_2 J_2^F \end{aligned}$$

While for vacancies, we have

$$\begin{aligned} rJ_1^V &= -\gamma_1 + q(\theta_1)(J_1^F - J_1^V) \\ rJ_2^V &= -\gamma_2 + q(\theta_2)(J_2^F - J_2^V) \end{aligned}$$

Zero profit for opening jobs in both sectors implies

$$J_1^V = J_2^V = 0$$

Using this, we have the value of filled jobs in the two sectors

$$J_1^F = \frac{\gamma_1}{q(\theta_1)} \quad \text{and} \quad J_2^F = \frac{\gamma_2}{q(\theta_2)}$$

The worker side is also identical, especially since workers don't have a choice affecting their status. In particular,

$$\begin{aligned} rJ_1^U &= z + \theta_1 q(\theta_1)(J_1^E - J_1^U) \\ rJ_2^U &= z + \theta_2 q(\theta_2)(J_2^E - J_2^U) \end{aligned}$$

where we have assumed the unemployment benefit is equal for both groups (this is not important, what's important is that unemployment benefits are not proportional to equilibrium wages).

Finally, the value of being employed for the two types of workers are

$$rJ_i^E = w_i - s(J_i^E - J_i^U)$$

The structure of the equilibrium is similar to before, in particular the modified golden rule and the two wage equations are:

$$\begin{aligned} Af'(k) &= r + \delta \qquad \text{M.G.R.} \\ w_1 &= (1-\beta)z + \beta\left[Af(k) - (r+\delta)k + \theta_1\gamma_1\right] \\ w_2 &= (1-\beta)z + \delta\left[Af(k)\eta - (r+\delta)k\eta + \theta_2\gamma_2\right] \end{aligned}$$

The most important result here is that wage differences between skilled unskilled workers are compressed.

To illustrate this, let us take a simple case and suppose first that

$$\gamma_1 = \gamma_2,\ n_1 = n_2,\ s_1 = s_2,\ z = 0.$$

Thus there are no differences in costs of creating vacancies, separation rates, unemployment benefits, and population growth rates between skilled and unskilled workers.

Then we have

$$u_2 > u_1$$

Why? Let's see

$$\begin{aligned} J_1^F &= \frac{\gamma}{q(\theta_1)} \qquad \text{and} \qquad J_2^F = \frac{\gamma}{q(\theta_2)} \\ J_2^F &> J_1^F \Longrightarrow \theta_1 < \theta_2 \Longrightarrow u_1 > u_2. \end{aligned}$$

High skill jobs yield higher rents, so everything else equal firms will be keener to create these types of jobs, and the only thing that will equate their marginal profits is a slower rate of finding skilled workers, i.e., a lower rate of unemployment for skilled than unskilled workers

There are also other reasons for higher unemployment for unskilled workers.

Also, $s_1 > s_2$ but lately $n_1 < n_2$ so the recent fall in n_1 and increase in n_2 should have helped unskilled unemployment.

But $z \uparrow$ has more impact on unskilled wages.

$\eta \uparrow \Longrightarrow$ "skill-biased" technological change.

$$\Longrightarrow \quad u_1 = cst,\ w_1 = cst$$
$$u_2 \downarrow,\ w_2 \uparrow$$

A set of interesting effects happen when r are endogenous. What are they?

Suppose we have $\eta \uparrow$, this implies that demand for capital goes up, and this will increase the interest rate, i.e., $r \uparrow$

The increase in the interest rate will cause

$$u_1 \uparrow, w_1 \downarrow .$$

What about labor force participation? Can this model explain non-participation?

Suppose that workers have outside opportunities distributed in the population, and they decide to take these outside opportunities if the market is not attractive enough. Suppose that there are N_1 and N_2 unskilled and skilled workers in the population. Each unskilled worker has an outside option drawn from a distribution $G_1(v)$, while the same distribution is $G_2(v)$ for skilled workers. In summary:

$$\begin{array}{ll} G_1(v) & N_1: \text{ unskilled} \\ G_2(v) & N_2: \text{ skilled} \end{array}$$

Given v; the worker has a choice between J_i^U and v.

Clearly, only those unskilled workers with

$$J_1^U \geq v$$

will participate and only skilled workers with

$$J_2^U \geq v$$

(why are we using the values of unemployed workers and not employed workers?)

Since L_1 and L_2 are irrelevant to steady-state labor market equilibrium above (because of constant returns to scale), the equilibrium equations are unchanged. Then,

$$\begin{aligned} L_1 &= N_1 \int_0^{J_1^U} dG_1(v) \\ L_2 &= N_2 \int_0^{J_2^U} dG_2(v). \end{aligned}$$

$\eta \uparrow,\ r \uparrow \Longrightarrow u_1 \uparrow,\ w_1 \downarrow\ J_1^U \downarrow$

$\Longrightarrow$ unskilled participation falls. (consistent with Juhn-Murphy and Topel's findings on US labor markets in the 1980s).

But this mechanism requires an interest rate response. Is the interest rate higher in the '80s?

Alternative formulation: the skilled do the unskilled jobs and there are not so many jobs (demand??). This takes us the next topic.

CHAPTER 12

Composition of Jobs

Search models, and more generally models with frictional labor markets, also provided a useful perspective for thinking about the endogenous composition of jobs. The "composition of jobs" here refers to the quality distribution of jobs, for example, some jobs may involve higher quality or newer vintage machines or more physical capital, and the same worker will be more productive in these jobs than others with lower quality machines or less physical capital. An investigation of the composition of jobs is interesting in part because this is one of the main margins in which labor markets may have different degrees of success in achieving and efficient allocation. For example, depending on labor market institutions or other features of the environment, the equilibrium may or may not involve the "appropriate" allocation of workers to firms, or the creation of the right types of jobs.

1. Endogenous Composition of Jobs with Homogeneous Workers

Let us start with the simplest setup, in which workers are homogeneous, but they can be employed in two different types of jobs. Labor and capital are used to produce two non-storable intermediate goods that are then sold in a competitive market and immediately transformed into the final consumption good. Preferences of all agents are defined over the final consumption good alone. Let us normalize the price of the final good to 1.

There is a continuum of identical workers with measure normalized to 1. All workers are infinitely lived and risk-neutral. They derive utility from the consumption of the unique final good and maximize the present discounted value of their utility. Time is continuous and the discount rate of workers is equal to r. On the other side of the market, there is a larger continuum of firms that are also risk-neutral with discount rate r.

The technology of production for the final good is:

$$Y = \left(\alpha Y_b^\rho + (1-\alpha) Y_g^\rho\right)^{1/\rho} \tag{12.1}$$

where Y_g is the aggregate production of the first input, and Y_b is the aggregate production of the second input, and $\rho < 1$. The elasticity of substitution between Y_g and Y_b is $1/(1-\rho)$ and α parameterizes the relative importance of Y_b. The subscripts g and b refer "good" and "bad" jobs as it will become clear shortly. This formulation captures the idea that there is some need for diversity in overall consumption/production, and is also equivalent to assuming that (12.1) is the utility function defined over the two goods.

Since the two intermediate goods are sold in competitive markets, their prices are:

$$\begin{aligned} p_b &= \alpha Y_b^{\rho-1} Y^{1-\rho} \\ p_g &= (1-\alpha) Y_g^{\rho-1} Y^{1-\rho} \end{aligned} \tag{12.2}$$

The technology of production for the inputs is Leontieff. When matched with a firm with the necessary equipment (capital k_b or k_g), a worker produces 1 unit of the respective good. The equipment required to produce the first input costs k_g while the cost of equipment for the second input is k_b. Let us assume that

$$k_g > k_b.$$

Before we move to the search economy, it is useful to consider the perfectly competitive benchmark. Since $k_g > k_b$, in equilibrium, we will have

$$p_g > p_b.$$

But firms hire workers at the common wage, w, irrespective of their sector. Thus, there will be neither wage differences nor bad nor good jobs. Also, since the first welfare theorem applies to this economy, the composition of output will be optimal.

Given the setup so far we can obtain the main idea before presenting the detailed analysis. As soon as we enter the world of search, there will be some rent-sharing. This implies that a worker who produces a higher valued output will receive a higher wage. As noted above, because $k_g > k_b$, the input which costs more to produce will command a higher price, thus in equilibrium $p_g > p_b$. Rent-sharing, then, leads to equilibrium wage differentials across identical workers. That is, $w_g > w_b$. Hence, the terms *good* and *bad* jobs. Next, it is intuitive that since, compared to the economy with competitive labor markets, good jobs have higher relative labor costs, their relative production will be less than optimal. In other words, the proportion of good (high-wage) jobs will be too low compared to what a social planner would choose. The rest of this section will formally analyze the search economy and establish these claims. It will then demonstrate that higher minimum wages and more generous unemployment benefits will improve the composition of jobs and possibly welfare.

1.1. The Technology of Search. As in the canonical search model, firms and workers come together via a matching technology $M(u, v)$ where u is the unemployment rate, and v is the vacancy rate (the number of vacancies). Once again, we assume that search is *undirected*, thus both types of vacancies have the same probability of meeting workers, and it is the total number of vacancies that enters the matching function. $M(u, v)$ is twice differentiable and increasing in its arguments and exhibits constant returns to scale. This enables me to write the flow rate of match for a vacancy as

$$\frac{M(u, v)}{v} = q(\theta),$$

where $q(.)$ is a differentiable decreasing function and

$$\theta = \frac{v}{u}$$

is the tightness of the labor market. It also immediately follows from the constant returns to scale assumption that the flow rate of match for an unemployed worker is

$$\frac{M(u, v)}{u} = \theta q(\theta).$$

In general, $q(\theta)$, $\theta q(\theta) < \infty$, thus it takes time for workers and firms to find suitable production partners. We also make the standard Inada-type assumptions on $M(u, v)$ which ensure that $\theta q(\theta)$ is increasing in θ, and that $\lim_{\theta\to\infty} q(\theta) = 0$, $\lim_{\theta\to 0} q(\theta) = \infty$, $\lim_{\theta\to\infty} q(\theta)\theta = 0$ and $\lim_{\theta\to 0} q(\theta)\theta = \infty$.

All jobs end at the exogenous flow rate s, and in this case, the firm becomes an unfilled vacancy and the worker becomes unemployed. Finally, there is free entry into both good and bad job vacancies, therefore both types of vacancies should expect zero net profits.

Let us denote the flow return from unemployment by z which will be thought as the level of unemployment benefit financed by lump-sum taxation. As usual, we assume that wages are determined by asymmetric Nash Bargaining where the worker has bargaining power β. Nash Bargaining per se is not essential, though rent-sharing is crucial for the results.

Firms can choose either one of two types of vacancies: *(i)* a vacancy for a intermediate good 1 - a *good job*; *(ii)* a vacancy for an intermediate good 2 - *a bad job*. Therefore, before opening a vacancy a firm has to decide which input it will produce, and at this point, it will have to buy the equipment that costs either k_b or k_g. The important aspect is that these *creation* costs are incurred before the firm meets its employees; this is a reasonable assumption, since, in practice, k corresponds to the costs of machinery, which are sector and occupation specific.

1.2. The Basic Bellman Equations. As usual, we will solve the model via a series of Bellman equations. We denote the discounted value of a vacancy by J^V, of a filled job by J^F, of being unemployed by J^U and of being employed by J^E. We will use subscripts b and g to denote good and bad jobs. We also denote the proportion of bad job vacancies among all vacancies by ϕ. Then, in steady state:

$$rJ^U = z + \theta q(\theta) \left[\phi J_b^E + (1-\phi) J_g^E - J^U\right] \tag{12.3}$$

Being unemployed is similar to holding an asset; this asset pays a dividend of z, the unemployment benefit, and has a probability $\theta q(\theta)\phi$ of being transformed into a bad job in which case the worker obtains J_b^E, the asset value of being employed in a bad job, and loses J^U; it also has a probability $\theta q(\theta)(1-\phi)$ of being transformed into a good job, yielding a capital gain $J_g^E - J^U$ (out of steady state, $\dot{J}^U$ has to be added to the right-hand side to capture future changes in the value of unemployment). Observe that this equation is written under the implicit assumption that workers will not turn down jobs, which we will discuss further below. The steady state discounted present value of employment can be written as:

$$rJ_i^E = w_i + s(J^U - J_i^E) \tag{12.4}$$

for $i = b, g$. (12.4) has a similar intuition to (12.3).

Similarly, when matched, both vacancies produce 1 unit of their goods, so:

$$rJ_i^F = p_i - w_i + s\left(J_i^V - J_i^F\right) \tag{12.5}$$

$$rJ_i^V = q(\theta)\left(J_i^F - J_i^V\right) \tag{12.6}$$

for $i = b, g$, where we have ignored the possibility of voluntary job destruction which will never take place in steady state.

Since workers and firms are risk-neutral and have the same discount rate, Nash Bargaining implies that w_b and w_g will be chosen so that:

$$\begin{aligned} (1-\beta)(J_b^E - J^U) &= \beta(J_b^F - J_b^V) \\ (1-\beta)(J_g^E - J^U) &= \beta(J_g^F - J_g^V) \end{aligned} \tag{12.7}$$

Note that an important feature is already incorporated in these expressions: workers cannot pay to be employed in high wage jobs: due to search frictions, at the moment a worker finds a job, there is bilateral monopoly, and this leads to rent-sharing over the surplus of the match.

As there is free-entry on the firm side, it should not be possible for an additional vacancy to open and make expected net profits. Hence:

$$J_i^V = k_i. \tag{12.8}$$

Finally, the steady state unemployment rate is given by equating flows out of unemployment to the number of destroyed jobs. Thus:

$$u = \frac{s}{s + \theta q(\theta)}. \tag{12.9}$$

1.3. Characterization of Steady State Equilibria. A steady state equilibrium is defined as a proportion ϕ of bad jobs, tightness of the labor market θ, value functions $J_b^V, J_b^F, J_b^E, J_g^V, J_g^F, J_g^E$ and J^U, prices for the two goods, p_b and p_g such that equations (12.2), (12.3), and (12.4), (12.5), (12.6), (12.7) and (12.8) for both $i = b$ and g are satisfied. The steady state unemployment rate is then given by (12.9).[1]

In steady state, both types of vacancies meet workers at the same rate, and in equilibrium workers accept both types of jobs, therefore $Y_b = (1-u)\phi$ and $Y_g = (1-u)(1-\phi)$. Then, from (12.2), the prices of the two inputs can be written as:

$$\begin{aligned} p_g &= (1-\alpha)(1-\phi)^{\rho-1}\left[\alpha\phi^\rho + (1-\alpha)(1-\phi)^\rho\right]^{\frac{1-\rho}{\rho}} \\ p_b &= \alpha\phi^{\rho-1}\left[\alpha\phi^\rho + (1-\alpha)(1-\phi)^\rho\right]^{\frac{1-\rho}{\rho}}. \end{aligned} \tag{12.10}$$

Simple algebra using (12.4), (12.5), (12.7) and (12.8) gives:

$$w_i = \beta\left(p_i - rk_i\right) + (1-\beta)rJ^U \tag{12.11}$$

as the wage equation. Intuitively, the surplus that the firm gets is equal to the value of output which is p_i minus the flow cost of the equipment, rk_i. The worker gets a share β of this, plus $(1-\beta)$ times his outside option, rJ^U. Using (12.5) and (12.6), the zero-profit condition (12.8) can be rewritten as:

$$\frac{q(\theta)(1-\beta)\left(p_b - rJ^U\right)}{r + s + (1-\beta)q(\theta)} = rk_b \tag{12.12}$$

$$\frac{q(\theta)(1-\beta)\left(p_g - rJ^U\right)}{r + s + (1-\beta)q(\theta)} = rk_g. \tag{12.13}$$

[1] One might wonder at this point whether a different type of equilibrium, with $J^U = J_b^E$ and workers accepting bad jobs with probability $\zeta < 1$, could exist. The answer is no. From equation (8.1), this would imply $J_b^V = J_b^F$, but in this case, firms could never recover their upfront investment costs.

A firm buys equipment that costs k_i, which remains idle for a while due to search frictions (i.e. because $q(\theta) < \infty$). This cost is larger for firms that buy more expensive equipment and open good jobs. They need to recover these costs in the form of a higher net flow profits: i.e. $p_g - rk_g > p_b - rk_b$. From rent-sharing, this immediately implies that $w_g > w_b$. More specifically, combining (12.11), (12.12) and (12.13), we get :

$$w_g - w_b = \frac{(r+s)\beta(rk_g - rk_b)}{(1-\beta)\,q(\theta)} > 0 \tag{12.14}$$

Therefore, wage differences are related to the differences in capital costs and also to the average duration of a vacancy. In particular, when $q(\theta) \to \infty$, the equilibrium converges to the Walrasian limit point, and both w_g and w_b converge to rJ^U, so wage differences disappear. The reason is that in this limit point, capital investments never remain idle, thus good jobs do not need to make higher net flow profits. Also, with equal creation costs, i.e., $k_b = k_g$, wage differentials disappear again.

Finally, (12.3) gives the value of an unemployed worker as

$$rJ^U = G(\theta,\phi) \equiv \frac{(r+s)z + \beta\theta q(\theta)\left[\phi(p_b - rk_b) + (1-\phi)(p_g - rk_g)\right]}{r+s+\beta\theta q(\theta)} \tag{12.15}$$

It can easily be verified that $G(.,.)$ is continuous, strictly increasing in θ, and strictly decreasing in ϕ. Intuitively, as the tightness of the labor market, θ, increases, workers find jobs faster, thus rJ^U is higher. Also as ϕ decreases, the greater fraction of good jobs among vacancies increases the value of being unemployed since $w_g > w_b$ (i.e., $J_g^V > J_b^E$). The dependence of rJ^U on ϕ is the general equilibrium effect mentioned in the introduction: as the composition of jobs changes, the option value of being unemployed also changes.

A steady-state equilibrium is characterized by the intersection of two loci: *bad job locus,* (12.12), and the *good job locus,* (12.13) (both evaluated with (12.10) and (12.15) substituted in).

The next figure draws these two loci in the θ-ϕ plane.

In this figure, the curve for (12.13), along which a firm that opens a good job vacancy makes zero-profits, is upward sloping: a higher value of ϕ increases the

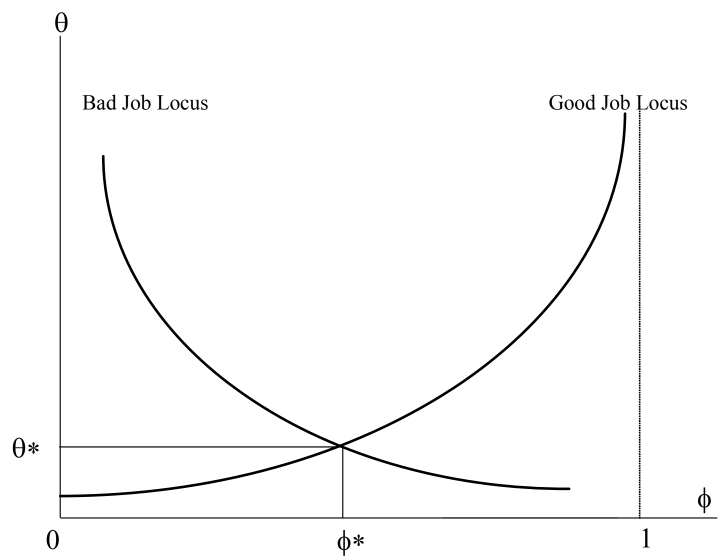

FIGURE 12.1

left hand side, thus θ needs to change to increase the right-hand side (and reduce the left-hand side through $G(\theta, \phi)$). Intuitively, an increase in ϕ implies a higher p_g (from equation (12.10)). So to ensure zero profits, θ needs to increase to raise the duration of vacancies. In contrast, (12.12) cannot be shown to be decreasing everywhere. Intuitively, an increase in ϕ reduces p_b, thus requires a fall in θ to equilibrate the market, but the general equilibrium effect through J^U (i.e. that a fall in ϕ reduces J^U) counteracts this and may dominate. This issue is discussed further below.

Here, let us start with the case in which $\rho \leq 0$, so that good and bad jobs are gross complements. In this case, it is straightforward to see that as ϕ tends to 1, (12.12) gives $\theta \to \infty$ whereas (12.13) implies $\theta \to 0$. Thus, the bad job locus is above the good job locus. The opposite is the case as ϕ goes to zero. Then by the continuity of the two functions, they must intersect at least once in the range $\phi \in (0, 1)$. Therefore, we can conclude that there always exists a steady state equilibrium with $\phi \in (0, 1)$ always exists and is characterized by (12.10), (12.11), (12.12), (12.13) and (12.15). In equilibrium, for all $k_g > k_b$, we have $p_g > p_b$ and $w_g > w_b$.

When $\rho > 0$, an equilibrium continues to exist, but does not need to be interior, so one of (12.12) and (12.13) may not hold. We now discuss a particular case of this.

1.4. Multiple equilibria. Since (12.12) can be upward sloping over some range, more than one intersections, hence multiple equilibria, are possible. (12.12) is more likely to be upward sloping when relative prices change little as a result of a change in the composition of jobs. Therefore, to illustrate the possibility of multiple equilibria, let us consider the extreme case where $\rho = 1$, so that goods g and b are perfect substitutes, and there are no relative price effects. Furthermore, we assume that

$$1 - 2\alpha > r(k_g - k_b).$$

In the absence of this assumption, good jobs are not productive enough, and will never exist in equilibrium.

The absence of substitution between good and bad jobs immediately implies that

$$p_g = 1 - \alpha > p_b = \alpha.$$

The equilibrium can then be characterized diagrammatically. To do this, totally differentiate (12.12) and (12.13), with $p_g = 1 - \alpha$ and $p_b = \alpha$, which gives

$$\left.\frac{d\theta}{d\phi}\right|_i = \frac{-\frac{\partial G(\theta,\phi)}{\partial \phi}}{\frac{\partial G(\theta,\phi)}{\partial \theta} - k_i \frac{(r+s)(1-\beta)q'(\theta)}{(1-\beta)q(\theta)^2}\frac{\partial G(\theta,\phi)}{\partial \theta}} > 0 \tag{12.16}$$

where $i = b$ is zero profit condition for bad jobs, (12.12), and $i = g$ is the zero profit condition for good jobs, (12.13). The derivative in (12.16) is positive, irrespective of whether it is for good or bad jobs, because $rJ^U = G(\theta, \phi)$ is decreasing in ϕ and increasing in θ, while $q'(\theta) < 0$. Since $k_b < k_g$, this equation also immediately implies that (12.12) is steeper than (12.13). So (12.12) has to intersect (12.13) from below if at all, in which case there will be three equilibria. This is shown in the next figure.

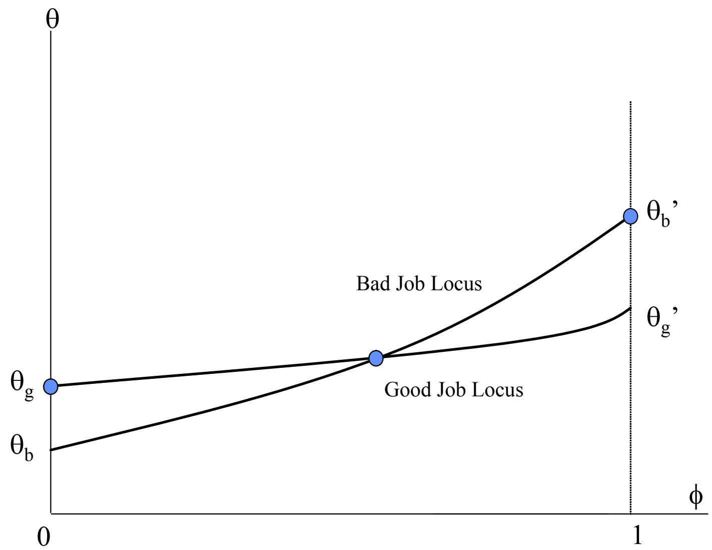

FIGURE 12.2

The first is a "mixed strategy" equilibrium at the point where the two curves intersect. The other two equilibria are more interesting. When $\phi = 0$, we have $\theta_g > \theta_b$, so that it is more profitable to open a good job. Hence there is an equilibrium in which all firms open good jobs. It is not profitable for firms to open a bad job, because when $\phi = 0$, workers receive high wages and have attractive outside options; so a firm that opens a bad job will be forced to pay a relatively high wage, making a deviation to a bad job unprofitable. In contrast, at $\phi = 1$, we have $\theta'_g < \theta'_b$, so it is an equilibrium for all firms to open bad jobs.

Intuitively, when all firms open bad jobs, the outside option of workers is low, so firms bargain to low wages, making entry relatively profitable. In equilibrium, θ has to be high to ensure zero profits. But a tight labor market (a high θ) hurts good jobs relatively more since they have to make larger upfront investments. The multiplicity of equilibria in this model illustrates the strength of the general equilibrium forces that operate through the impact of job composition on the overall level of wages.

1.5. Welfare. Let us next analyze the welfare properties of equilibrium using the notion of total surplus as in the baseline search model. In this case, total surplus (in steady state) can be written as:

$$TS = (1-u)\left[\phi(p_b - rk_b) + (1-\phi)(p_g - rk_g)\right] - \theta u\left(\phi rk_b + (1-\phi)rk_g\right) \tag{12.17}$$

Total surplus is equal to total flow of net output, which consists of the number of workers in good jobs ($(1-\phi)(1-u)$) times their net output (p_g minus the flow cost of capital rk_g), plus the number of workers in bad jobs ($\phi(1-u)$) times their net product ($p_b - rk_b$), minus the flow costs of job creation for good and bad vacancies (respectively, $\theta u(1-\phi)rk_g$ and $\theta u\phi rk_b$).

It is straightforward to locate the set of allocations that maximize total social surplus. This set would be the solution to the maximization of (12.17) subject to (12.9). Inspecting the first-order conditions of this problem, it can be seen that decentralized equilibria will not in general belong to this set, thus a social planner can improve over the equilibrium allocation. The results regarding the socially optimal amount of job creation are standard: if β is too high, that is $\beta > \eta(\theta)$ where $\eta(\theta)$ is elasticity of the matching function, $q(\theta)$, then there will be too little job creation, and if $\beta < \eta(\theta)$, there will be too much. Since this paper is concerned with the composition of jobs, we will not discuss these issues in detail. Instead, we will show that irrespective of the value of θ, the equilibrium value of ϕ is always too high; that is, there are too many bad jobs relative to the number of good jobs.

To prove this claim, it is sufficient to consider the derivative of TS with respect to ϕ at $z = 0$ (note the constraint, (12.9), does not depend on ϕ):

$$\frac{dTS}{d\phi} = (1-u)\cdot\left[\frac{d(\phi p_b + (1-\phi)p_g)}{d\phi}\right] - (1-u+u\theta)\cdot\{rk_b - rk_g\} \tag{12.18}$$

For the composition of jobs to be efficient at the laissez-faire equilibrium, (12.18) needs to equal zero when evaluated in the equilibrium characterized above. Some simple algebra using (12.9), (12.10), (12.12) and (12.13) to substitute out u, and k_i

gives (details of the algebra available upon request):

$$\left.\frac{dTS}{d\phi}\right|_{dec.\ eq.} = \frac{\theta q(\theta)}{s+\theta q(\theta)} \cdot \left(1 + \frac{(s+q(\theta))(1-\beta)}{r+s+(1-\beta)q(\theta)}\right) \cdot (p_b - p_g) < 0$$

This expression is always negative, irrespective of the value of θ, so starting from laissez-faire equilibrium, a reduction in ϕ will increase social surplus. Therefore, we can conclude that, given the labor market tightness θ, a surplus-maximizing social planner would choose $\phi^s(\theta) < \phi^*(\theta)$, where $\phi^*(\theta)$ is the decentralized equilibrium with $z = 0$. In other words, the equilibrium proportion of bad jobs is too high.

The intuition is simple; in a decentralized equilibrium, it is always the case that $w_g > w_b$. Yet, firms do not take into account the higher utility they provide to workers by creating a good job rather than a bad job, hence there is an uninternalized positive externality, which leads to an excessively high fraction of bad jobs in equilibrium. Search and rent-sharing are crucial for this result. Search ensures that firms have to share the ex post rents with the workers, and they cannot induce competition among workers to bid down wages. Firms would ideally like to contract with their workers on the wage rate before they make the investment decision, but search also implies that they do not know who these workers will be, thus cannot contract with them at the time of investment.

1.6. The Impact of Minimum Wages and Unemployment Benefits. As is usual in models with potential multiple equilibria, only the comparative statics of "extremal" equilibria are of interest. Therefore, let us focus on an economy where in equilibrium (12.13) cuts (12.12) from below (or alternatively, an economy with a unique equilibrium). Now consider an increase in z which corresponds to the UI system becoming more generous. Both the bad job locus, (12.12), and the good job locus, (12.13), will shift down. Hence, θ will definitely fall. It is also straightforward to verify that (12.12) will shift by more, therefore, ϕ is unambiguously reduced. Intuitively, with ϕ unchanged, relative prices and hence wages will be unchanged, but then with the higher unemployment benefits, workers would prefer to wait for good jobs rather than accept bad jobs. This increases w_b and reduces ϕ (the fraction of bad jobs).

Furthermore, a more generous unemployment benefit not only increases the fraction of good jobs, but may also increase the total number of good jobs. Totally differentiating (12.12) and (12.13), we obtain that the total number of good jobs will increase if and only if:

$$w_g - w_b > \left(\frac{1}{\eta(\theta)} - 1\right) u(1-\phi)\left(\frac{d(p_g - p_b)}{d\phi}\right)$$

where recall that $\eta(\theta)$ is the elasticity of $q(\theta)$. This inequality is likely to be satisfied when the two inputs are highly substitutable, i.e. ρ close to 1; when wage differences are large; when $\eta(\theta)$ is close to 1; and/or when unemployment is low to start with. Thus, it is only increases in unemployment benefit starting from moderate levels that increase the number of good jobs.

The impact on welfare depends on how large the effect on θ is relative to the effect on ϕ. We can see this by totally differentiating (12.17) after substituting for u. This gives a relationship between θ and ϕ, drawn as the dashed line in the next figure, along which total surplus is constant.

Shifts of this curve towards North-East give higher surplus. When this curve is steeper than (12.13), a higher z can improve welfare, and this is the case drawn in the figure. For example, if β is very low to start with, then unemployment will be too low relative to the social optimum, and in this case an increase in z will unambiguously increase total welfare.

More generally, irrespective of whether total surplus increases, a more generous unemployment benefit raises average labor productivity, $\phi p_b + (1-\phi)p_g$, which is unambiguously decreasing in ϕ. Therefore, when unemployment benefits increase, the composition of jobs shifts towards more capital intensive good jobs, and labor productivity increases.

A minimum wage has a similar effect on job composition. Consider a minimum wage $\underline{w}$ such that $w_b < \underline{w} < w_g$, so it is only binding for bad jobs. The equation for

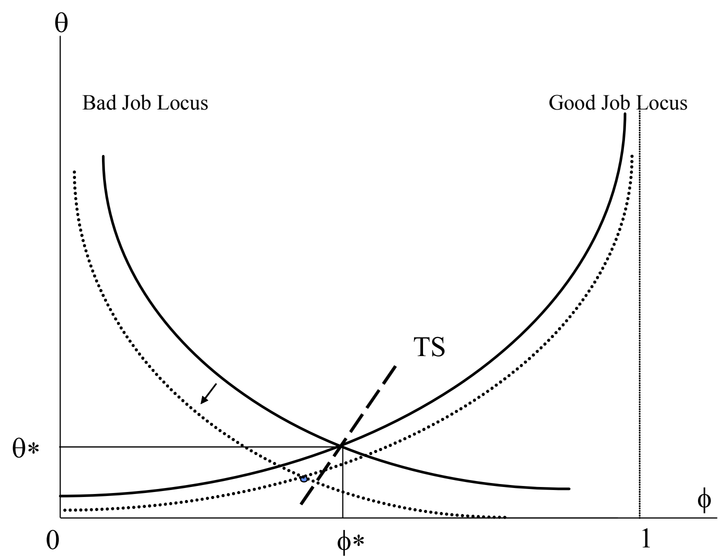

FIGURE 12.3

J_b^F now becomes:

$$J_b^F = \frac{p_b - \underline{w} + sk_b}{r + s}.$$

Then, (12.12) changes to:

$$q(\theta)\frac{p_b - \underline{w}}{r + s + q(\theta)} = rk_b. \tag{12.19}$$

Since at a given θ, the left-hand side of (12.19) is less than that of (12.12), the impact of higher minimum wages is to shift the bad job locus, curve (12.12), down. The good job locus is still given by (12.13), but now, combining (12.3) and (12.4),

$$rJ^U = G(\theta, \phi) \equiv \frac{(r+s)z + \beta\theta q(\theta)\left[\phi\underline{w} + (1-\phi)(p_g - rk_g)\right]}{r + s + \theta q(\theta)(1 - (1-\beta)(1-\phi))}$$

Since $\underline{w} > w_b$, both curves shift down, but as in the case of unemployment benefits, (12.12) shifts down by more, so both ϕ and θ fall. Again, the rise in minimum wages can increase the number, not just the proportion, of good jobs and total welfare. Moreover, for the same decline in θ, an increase in minimum wages reduces ϕ more than an increase in z, therefore, minimum wages appear to be more powerful in shifting the composition of employment away from bad towards good jobs.

Overall, we can conclude that both the introduction of a minimum wage $\underline{w}$ and an increase in unemployment benefit z decrease θ and ϕ. Therefore, they improve the composition of jobs and average labor productivity, but increase unemployment. The impact on overall surplus is ambiguous.

2. Endogenous Composition of Jobs with Heterogeneous Workers

Now consider a somewhat more realistic environment in which workers are also of heterogeneous skills. In particular, consider a world in which workers may have high or low skills and they have to match with firms. Firms will choose the level of their capital stock before matching with the workers. The basic idea that will be highlighted by the model is that when either the productivity gap between skilled and unskilled workers is limited or when the number of skilled workers in the labor force is small, it will be profitable for firms to create jobs that to employ both skilled and unskilled workers. But when the productivity gap is large or that are a sufficient number of skilled workers, it may become profitable for (some) firms to target skilled workers, designing the jobs specifically for these workers. Then these firms will wait for the skilled workers, and will try to screen the more skill once among the applicants. In the meantime, there will be lower-quality (low capital) jobs specifically targeted at the unskilled.

Suppose that there are two types of workers. The unskilled have human capital (productivity) 1, while the skilled have human capital $\eta > 1$. Denote the fraction of skilled workers in the labor force by ϕ.

Firms choose the capital stock k before they meet a worker, and matching is assumed to be random, in the sense that each firm, irrespective of its physical capital, has exactly the same probability of meeting different types of workers. Once the firm and the worker match, separating is costly, so there is a quasi-rent to be divided between the pair. Here, the economy is assumed to last for one period, so if the firm and worker do not agree they lose all of the output (see Acemoglu, 1999, for the model where the economy is infinite-horizon and agents who do not agree with their partners can resample). Therefore, bargaining will result in workers receiving a certain fraction of output, which is again denoted by β.

The production function of a pair of worker and firm is

$$y = k^{1-\alpha} h^{\alpha},$$

where k is the physical capital of the firm and h is the human capital of the worker.

Firms choose their capital stock to maximize profits, before knowing which type of worker will apply to their job. For simplicity, we assume that firms do not bear the cost of capital if they decides not to produce with the worker who has applied to the job. We also denote the cost of capital by c.

Their expected profits are therefore given by

$$\phi x^{H} (1-\beta) \left(k^{1-\alpha}\eta - ck\right) + (1-\phi) x^{L} (1-\beta) \left(k^{1-\alpha} - ck\right),$$

where x^{j} is the probability, chosen by the firm, that it will produce with a worker of type j conditional on matching that type of worker. Therefore, the first term is profits conditional on matching with a skilled worker, and the second term gives the profits from matching with an unskilled worker.

There can be to different types of equilibria in this economy:

(1) A pooling equilibrium in which firms choose a level of capital and use it both of skilled and unskilled workers. We will see that in the pooling equilibrium inequality is limited.
(2) A separating equilibrium in which firms target the skilled and choose a higher level of capital. In this equilibrium inequality will be greater.

In this one-period economy, firms never specifically target the unskilled, but that outcome arises in the dynamic version of this economy.

Now it is straightforward to characterize the firms profit maximizing capital choice and the resulting organization of production (whether firms will employ both skilled and unskilled workers). It turns out that first choose the pooling strategy as long as

$$\eta < \left(\frac{1-\phi}{\phi^{\alpha}-\phi}\right)^{1/\alpha}$$

Therefore, a sufficiently large increase in η (in the relative productivity of skilled workers) and/or in ϕ (the fraction of skilled workers in the labor force) switches the economy from pooling to separating).

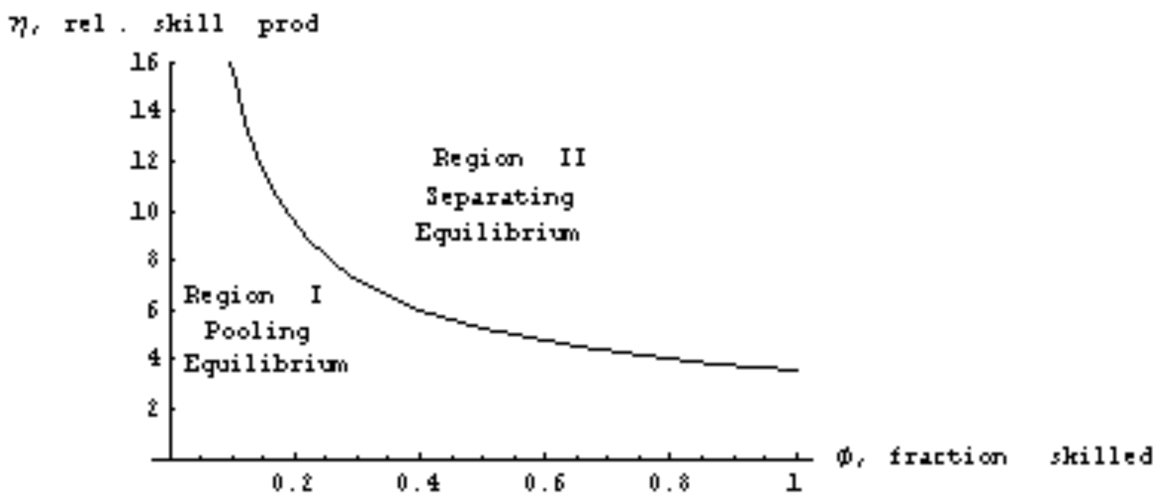

Figure 12.4

Such a switch will be associated with important changes in the organization of production, an increase in inequality, and a decline in the wages of low-skill workers.

Is there any evidence that there has been such a change in the organization of production? This is difficult to ascertain, but some evidence suggests that there may have been some important changes in how jobs are designed and organized now.

First, firms spend much more on recruiting, screening, and are now much less happy to hire low-skill workers for jobs that they can fill with high skill workers.

Second, as already mentioned above, the distribution of capital to labor across industries has become much more unequal over the past 25 years. This is consistent with a change in the organization of production where rather than choosing the same (or a similar) level of capital with both skilled and unskilled workers, now some firms target the skilled workers with high-capital jobs, while other firms go after unskilled workers with jobs with lower capital intensity.Third, evidence from

Figure 12.5

the CPS suggests that the distribution of jobs has changed significantly since the early 1980s, with job categories that used to pay "average wages" have declined in importance, and more jobs at the bottom and top of the wage distribution. In particular, if we classify industry-occupation cells into high-wage the middle-wage and low-wage ones (based either on wages or residual wages), there are many fewer workers employed in the middle-wage cells today as compared to the early 1980s, or the weight-at-the-tales of the vob quality distribution has increased substantially as the next figure shows.

This framework also suggests that there should be better "matching" between firms and workers now, since firms are targeting high skilled workers. Therefore,

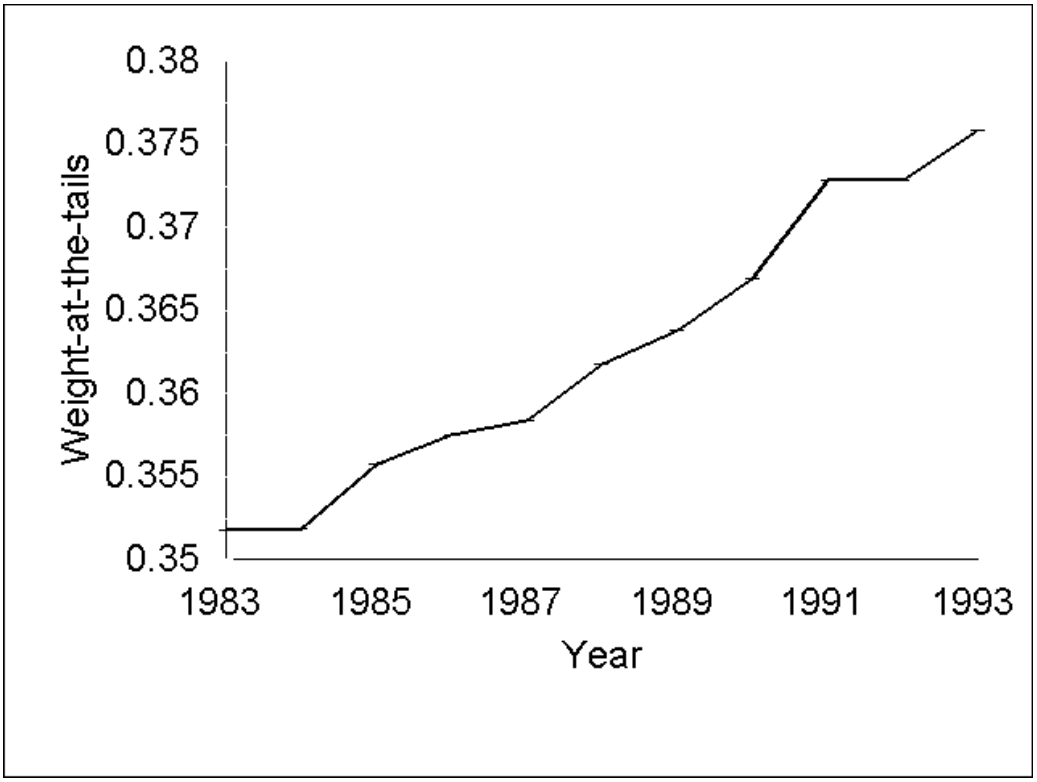

Figure 12.6. The evolution of the percentage of employment in the top and bottom 25 percentile industry-occupation cells (weight-at-the-tails of the job quality distribution).

measures of mismatch should have declined over the past 25 or so years. Consistent with this prediction, evidence from the PSID suggests that there is much less over- or under-education today than in the 1970s.

CHAPTER 13

Wage Posting and Directed Search

1. Inefficiency of Search Equilibria with Investments

Before turning to wage posting and directed search, let us highlight a more severe (and more fundamental) source of inefficiency in search models than the bargaining power not satisfying the Hosios condition. This results in the presence of *investments.*

Production still requires 1 firm - 1 worker, but now there is the intensive margin of capital per worker. In particular, this pair produces $f(k)$, where k is capital per worker. We assume

$$f' > 0, \quad f'' < 0$$

The most important feature is that k is to be chosen ex ante and is irreversible. The important economic implications of this are two:

(1) If there is bargaining, at this stage of bargaining, the capital is already sunk and the capital to labor ratio is irreversibly determined.
(2) While looking for a worker, the firm incurs an opportunity cost equal to be user cost of capital times the amount of capital that has, i.e., $u_k \times k$, where u_k is the user cost which will be determined below.

Trading frictions will be modeled in a way similar to before, but since my interest here is with "inefficiency," which is easily possible with increasing or decreasing returns to scale in the matching technology, I will assume constant returns to scale from the beginning. I will also develop the notation that will be useful when we look at wage posting and directed search.

First note that if $M = M(U, V)$ exhibits constant returns to scale, then exploiting the standard linear homogeneity properties, we can write

$$\begin{aligned} q &= \frac{M}{V} = M\left(\frac{U}{V}, 1\right) \\ &= q(\theta) \end{aligned}$$

where $\theta \equiv V/U$ is the tightness of the labor market (the vacancy to unemployment ratio), and the function $q(\theta)$ is decreasing in θ given our assumptions above. This means that vacancies have a harder time finding matches in a tighter labor market.

This is the standard notation in the Diamond-Mortensen-Pissarides macro search models.

Moreover,

$$\begin{aligned} p &= \frac{M}{U} = \frac{V}{U} M\left(\frac{U}{V}, 1\right) \\ &= \theta q(\theta) \end{aligned}$$

where $\theta q(\theta)$ is increasing in θ. This means that unemployed workers have an easier time finding matches in a tighter labor market.

Now let us develop a slightly different notation. Assume that if there are Q workers searching for 1 job (think of the analogy to *queues*), Q is equivalent to $1/\theta$ in the above notation.

Then with constant returns to scale, we have

$\mu(Q)$: flow rate of match for workers, assumed it is continuously differentiable and $\mu' < 0$

$\eta(Q) \equiv Q\mu(Q)$: flow rate of match for vacancy, with $\eta' > 0$

The fact that μ, η are simply functions of Q is equivalent to assuming Constant Returns to Scale.

As before let r be the rate of time preference, and s be the separation rate due to destruction of capital

Here let us change the order a little, and start with the efficient allocation, which is again a solution to the planner's problem subject to the search constraints.

The objective function of the planner can be written as:

$$\int_0^{\infty} e^{-rt}\left[\underbrace{\left(\mu(Q_t)\frac{f(k_t)-(r+s)k_t}{r+s}\right)}_{\text{net output of a matched worker}}u_t - \underbrace{(r+s)k_t\frac{u_t}{Q_t}}_{\text{cost of unfilled vacancies}}\right]dt$$

where u_t is the measure of unemployed workers, or alternatively the unemployment rate, at time t.

Here it is easy to see that $(r+s)k$ is the flow cost of investment, or user cost of capital, k. (k paid up front and rk opportunity cost, sk cost of destruction). The planner incurs this cost for $V_t = u_t/Q_t$ vacancies

Less obvious at first, but equally intuitive is that the value of an unemployed worker is that with probability $\mu(Q_t)$ he will find a job, in which case he will produce a net output of $f(k_t) - (r+s)k_t$, until the job is destroyed, which has discounted value $\frac{f(k_t)-(r+s)k_t}{r+s}$, thus the value of an unemployed worker is

$$\mu(Q_t)\frac{f(k_t)-(r+s)k_t}{r+s}.$$

This expression already imposes that all firms will choose the same capital level, and no segmentation in the market (Homework exercise: set up and solve this problem when the planner allows firms to choose different levels of capital).

The constraint that the planner faces is very similar to the flow constraints we saw above:

$$\dot{u}_t = s(1-u_t) - \mu(Q_t)u_t$$

This equation says that the evolution of unemployment is given by the flows into unemployment, $s(1-u_t)$, and exits from unemployment, i.e., job creation, $\mu(Q_t)u_t$.

Now we can write the Current Value Hamiltonian as

$$H(k,Q,u,\lambda) = u\left[\mu(Q)\left(\frac{f(k)}{r+s} - k\right) - \frac{(r+s)k}{Q}\right] + \lambda\left[s(1-u) - \mu(Q)u\right]$$

The necessary conditions are

$$H_k = u\left(\mu(Q)\left(\frac{f'(k)}{r+s}-1\right)-\frac{(r+s)}{Q}\right)=0$$
$$H_Q = u\left(\mu'(Q)\left(\frac{f(k)}{r+s}-k-\lambda\right)+\frac{(r+s)}{Q^2}k\right)=0$$
$$H_u = \mu(Q)\left(\frac{f(k)}{r+s}-k\right)-\frac{(r+s)}{Q}k-\lambda(s+\mu(Q))=r\lambda-\dot{\lambda}$$

Again, focusing on steady state, we impose

$$\dot{\lambda}=0$$
$$H_u=r\lambda \Longrightarrow \lambda=\frac{\mu(Q)\left(\frac{f(k)}{r+s}-k\right)-\frac{(r+s)}{Q}k}{r+s+\mu(Q)}$$

which is the shadow value of an unemployed worker. This equation has a very intuitive interpretation. The shadow value of a worker is given by the probability (flow rate) that he will create a job, which is $\mu(Q)$, and the value of the job is

$$\left(\frac{f(k)}{r+s}-k\right).$$

While unemployed, the worker induces the planner to have more vacancies open (so as to keep Q constant), hence the term

$$-\frac{(r+s)}{Q}k.$$

Finally, once the job is destroyed, which happens at the rate s, a new cycle begins, at the rate $\mu(Q)$, which gives the denominator for discounting.

The condition that $H_k=0$ gives

$$\Longrightarrow \frac{Q^S\mu(Q^S)f'(k^s)}{(r+s)(r+s+Q^S\mu(Q^S))}=1 \tag{13.1}$$

Now combining this and the value of λ obtained about with $H_u=0 \Longrightarrow$

$$f(Q^S)\frac{\mu'(Q^S)}{r+s}+\frac{r+s+\mu(Q^S)+Q^S\mu'(Q^S)-(Q^S)^2\mu'(Q^S)}{(Q^S)^2}k=0 \tag{13.2}$$

Conditions (13.1) and (13.2) characterize the constrained efficient allocation.

Next, consider the equilibrium allocation. With bargaining this corresponds to:

$$rJ^F(k) = f(k)-w(k)-sJ^F(k)$$
$$rJ^V(k) = \eta(Q)(J^F(k)-J^V(k))-sJ^V(k)$$

Recall that there is random matching, so Q workers for each vacancy. Then I can write

$$\begin{aligned} rJ^E(k) &= w(k) + s(J^U - J^E(k)) \\ rJ^U &= \mu(Q)\int a(k)(J^E(k) - J^U)dF(k) \end{aligned}$$

where $a(k)$ is the decision rule of the worker on whether to match with a firm with capital k, and $F(k)$ is the endogenous distribution of capital (please do not confuse this with f which is the production function).

Nash Bargaining again implies:

$$(1-\beta)(J^E(k) - J^U) = \beta(J^F(k) - J^V(k))$$

Now we will impose free entry as in the basic Mortensen-Pissarides models, so

$$J^V(k) - k = 0$$

That is, opening a job costs k (the sunk investment), and has a return of $J^V(k)$.

$$\Longrightarrow w(k) = \beta\left(f(k) - (r+s)k\right) + (1-\beta)rJ^U$$

Now use this wage rule with J^V and J^F

$$J^V(k) = \frac{\eta(Q)\left((1-\beta)f(k) + \beta\left(r+s\right)k - (1-\beta)rJ^U\right)}{(r+s)(r+s+\eta(Q))} \tag{13.3}$$

Also recall that $\eta(Q) = Q\mu\left(Q\right)$.

How is the capital-labor ratio chosen? Firms will clearly choose it to maximize profits: that is,

$$k \text{ maximizes } J^V(k) - k.$$

Since this is a strictly concave problem, this implies that all firms will choose the same level of capital, k^B

$\Longrightarrow$

$$F(k) \text{ is a degenerate distribution with all of its mass at } k^B$$

where

$$\frac{\eta(Q^B)(1-\beta)f'(k^B)}{(r+s)(r+s+(1-\beta)\eta(Q^B))} = 1 \tag{13.4}$$

with Q^B as the equilibrium queue length in the economy.

Now use (13.3) with J^V and J^E to obtain an equation determining Q^B.

$$\frac{\eta(Q^B)(1-\beta)f(k^B)}{r+s} = \left(r+s+(1-\beta)\eta(Q^B)+\beta\mu(Q^B)\right)k^B \tag{13.5}$$

The equations (13.4) and (13.5) characterize the equilibrium, and can be directly compared to the conditions (13.1) and (13.2) for the efficient allocation.

First, compare k^S to k^B: we can see that for all $\beta > 0$, $k^B < k^S$. In other words, there will be underinvestment as long as workers have ex post bargaining power. This is a form of **holdup**, in the sense that the firm makes an investment and the returns from the investments are shared between the worker and the firm. Because the investment is made before there is a match, there is no feasible way of contracting between the worker and the firm in order to avoid this holdup problem.

Thus the only way of obtaining efficiency is to set $\beta = 0$.

What about Q^S versus Q^B?

To compare Q^S versus Q^B, let $f(k^B) = f(k^S)$, then we obtain

$$\beta = \beta^*(Q) \equiv \frac{\eta'(Q)Q}{\eta(Q)} \equiv 1 + \frac{\mu'(Q)Q}{\mu(Q)},$$

is necessary and sufficient for $Q^S = Q^B$.

In other words, with $f(k^B) = f(k^S)$, we are back to the model without investment, so all we need is the Hosios condition for efficiency.

$$M = \mu \cdot U \Longrightarrow M_U = \mu' Q + \mu,$$
$$\Longrightarrow \frac{M_U U}{M} = 1 + \frac{\mu' Q}{\mu},$$

which can be verified as the Hosios condition in this case.

Thus when $f(k^B) = f(k^S)$, the Hosios condition is necessary and sufficient for efficiency.

This is not surprising, since with $f(k^B) = f(k^S)$, the economy is identical to the one with fixed capital.

The key question is whether it is possible to ensure both $f(k^B) = f(k^S)$ and $Q^S = Q^B$ simultaneously.

Of course, from the analysis the answer is no.

If $\beta > 0$, hold-up problem and $k^S > k^B$

If $\beta = 0$, the excessive entry of firms $Q^B < Q^S$.

THEOREM 13.1. *Constrained efficiency is impossible with ex ante investments and ex post bargaining.*

The intuition is quite straightforward: as long as $\beta > 0$, there is rent sharing on the marginal increase in productivity, thus hold-up. But $\beta = 0$ is inconsistent with optimal entry.

2. The Basic Model of Directed Search

Workers do not randomly search among all possible jobs, but apply for jobs that are more likely to be appropriate for their skills and interests. How do we model this? And how does this changed the positive and normative implications of search models?

One way is to construct the general equilibrium model with a non-degenerate wage distribution and then allow workers to search, perhaps in a smart way, among these jobs.. These models have the potential of leading to a coherent general equilibrium model with sequential search. But they are rather difficult to work with.

However, when all workers are assumed to observe all possible wage offers and can direct their search to one of these potential offers, then these models become quite tractable. At some level, this modeling assumption removes the actual "search" problem, but something akin to this, the coordination problem among the application decisions of workers is present in place the same role.

These models are sometimes referred to *competitive search* models, but is more useful to emphasize the two underlying assumptions: wage posting and directed search, so we will refer to them as *directed search* models.

To bring out the most important points, let us start from the economic environment of the search and investment model. Recall that in this model there are ex ante investments by firms, and bilateral search to form productive partnerships. In particular, recall that production requires 1 firm - 1 worker, with access to the production function $f(k)$, where k is capital for worker chosen before the matching stage by the firm. Recall that

$$f' > 0, \quad f'' < 0$$

The rate of time preference is r, and the rate of separation due to the destruction of capital by s.

We will now think of search frictions as equivalent to "coordination frictions". In particular, if there are an average of q workers per vacancy of a certain type then the flow rate of match for workers is $\mu(q)$, which is assumed to be continuously differentiable with $\mu' < 0$. Similarly, the flow rate of matching for a vacancy is $\eta(q) \equiv q\mu(q)$, where I am purposefully using the notation little q to distinguish this from the capital Q before which referred to the economy-wide queue length, whereas q it's specific to a type of job.

So this might seem somewhat strange; workers know what the various wages are, but conditional on applying to a job they may not get it; but this is sensible when there is no (centralized) coordination in the economy, because too many other people may be applying specifically to that job. The urn ball technology captured is in a very specific way, and in particular, we had

$$\eta(q) = 1 - \exp(-q) \text{ and } \mu(q) = \frac{1 - \exp(-q)}{q}$$

The technology here generalizes that.

As explained above, first all firms post wages w and also choose their capital k.

Workers observe *all* wages and then choose which job to seek. (they do not care about capital stocks).

Now more specifically let $q(w)$ be the ratio of workers seeking wage w to firms offering w. then $\mu(q(w))$ is flow rate of workers getting a job with wage w and $\eta(q(w))$ is flow rate of firms filling their jobs.

What equilibrium concept should we use here? Thinking about it intuitively, it is clear that we should ensure that workers apply to jobs that maximize utility and anticipate queue lengths at various wages rationally. This is straightforward.

The harder part is for firms. Firms should choose wages and investment to maximize profits, anticipating queue lengths at wages not offered in equilibrium. The last part is very important and corresponds to *Subgame perfection.* This is obviously important, since we have a dynamic economy, and you can see what will go wrong if we didn't impose subgame perfection.

Before we go further, let us first write the Bellman Equations, which are intuitive and standard for the firm (again imposing steady state throughout):

$$\begin{aligned} rJ^V(w,k) &= \eta(q(w))(J^F(w,k) - J^V(w,k)) - sJ^V(w,k) \\ rJ^F(w,k) &= f(k) - w - sJ^F(w,k) \end{aligned}$$

implying a simple equation for the value of firm

$$J^V(w,k) = \frac{\eta(f(k)-w)}{(r+s)(r+s+\eta)}$$

which we will use below.

The value of an employed worker is also simple:

$$rJ^E(w) = w + s(J^U - J^E(w))$$

What is slightly more involved is the value for unemployed worker.

Recall that unemployed workers take an important action: they decide which job to seek. Let $J^U(w)$ be the value of an unemployed worker when seeking wage w.

$$\underbrace{rJ^U(w)}_{\text{utility of applying to wage } w} = \mu(q(w))[J^E(w) - \underbrace{J^U}_{\substack{\text{maximal utility} \\ \text{of unemployment}}}]$$

where I have suppressed unemployment benefits without loss of any generality.

So what is J^U? Clearly:

$$J^U = \max_{w \in \mathcal{W}} J^U(w)$$

where $\mathcal{W}$ is the support of the equilibrium wage distribution.

Now this already builds in the requirement that w maximizes $J^U(w)$.

Also it is clear that w, k should maximize $J^V(w, k)$.

But what are the $q(w)$'s?

If we did not impose subgame perfection, then we could have crazy $q(w)$'s. Instead, firms would have to anticipate what workers would do if they deviate and create a new wage distribution.

So off-the-equilibrium path $q(w)$ should satisfy

$$\mu(q(w)) \left[J^E(w) - J^U \right] = rJ^U$$

or if $J^E(w) - J^U < rJ^U$, then $q(w) = 0$.

To define an equilibrium more formally, let an allocation be a tuple $\left\langle \mathcal{W}, Q, K, J^U \right\rangle$, where $\mathcal{W}$ is the support of the wage distribution, $Q : \mathcal{W} \to \mathbb{R}$ is a queue length function, $K : \mathcal{W} \rightrightarrows \mathbb{R}$ is a capital choice correspondence, and $J^U \in \mathbb{R}$ is the equilibrium utility of unemployed workers.

DEFINITION 13.1. *A directed search equilibrium satisfies*

(1) *For all* $w \in \mathcal{W}$ *and* $k \in K(w)$, $J^V(w, k) = 0$.
(2) *For all* k *and for all* w, $J^V(w, k) \leq 0$.
(3) $J^U = \sup_{w \in \mathcal{W}} J^U(w)$.
(4) $Q(w)$ *s.t.* $\forall w$, $J^U \geq J^U(w)$, *and* $Q(w) \geq 0$, *with complementary slackness.*

In words, the first condition requires firms to makes zero profits when they choose equilibrium wages and corresponding capital stocks. The second requires that for all other capital stock and wage combinations, profits are nonpositive. The third condition defines J^U as the maximal utility that an unemployed worker can get. The fourth condition is the most important one. It defines queue lengths to be such that workers are indifferent between applying to available jobs, or if they cannot be made indifferent, nobody applies to a particular job (thus the *complementary slackness* part is very important). This builds in the notion of *subgame perfection.*

Now we have

THEOREM 13.2. ***(Acemoglu and Shimer)*** *Equilibrium k, w, q maximize $\frac{\mu(q)w}{r+s+\mu}$ $(= rJ^U)$ subject to $\eta(q)\frac{(f(k)-w)}{r+s+\eta(q)} = (r+s)k$. And conversely, any solution to this maximization problem can be supported as an equilibrium.*

Basically what this theorem says is that the equilibrium will be such that the utility of an unemployed worker is maximized subject to zero profit.

PROOF. (sketch) Suppose not. Take k', w', q' which fails to maximize the above program. Then another firm can offer k'', w'' where (k^*, w^*, q^*) is the solution and $w'' = w^* - \varepsilon$. For ε small enough workers prefer k'', w'' to k', w', so $q'' > q^*$, which implies that k'', w'' makes positive profits, proving that (k', w', q) can't be an equilibrium. □

This theorem is very useful because it tells us that all we have to do is to solve the program:

$$\max \frac{\mu(q)w}{r+s+\mu(q)}$$
$$\text{s.t.} \quad \frac{\eta(q)(f(k)-w)}{r+s+\eta(q)} = (r+s)k$$

Is this a convex problem?

No, but let's assume differentiablity (which we have so far), then first order conditions are necessary.

Forming the Lagrangian with multiplier λ

$$\frac{\eta(q)f'(k)}{r+s+\eta(q)} = r+s \tag{13.6}$$

$$\frac{\mu(q)}{r+s+\mu(q)} - \frac{\lambda\eta(q)}{r+s+\eta(q)} = 0 \tag{13.7}$$

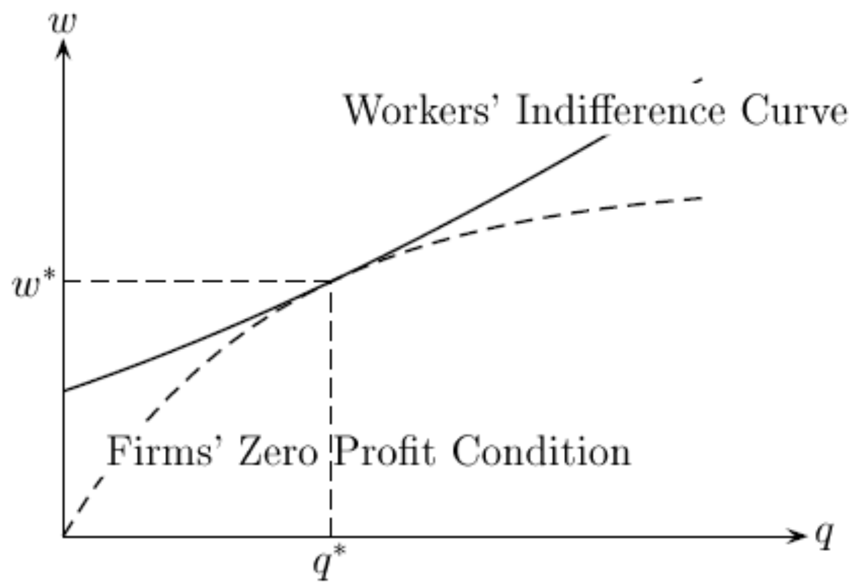

FIGURE 13.1

and

$$\frac{(r+s)\mu'(q)}{(r+s+\mu(q))^2}+\lambda\left(\frac{(r+s)\eta'(q)(f(k)-w)}{(r+s+\eta(q))^2}\right)=0 \tag{13.8}$$

Now (13.6) is identical to (13.1) above, which was

$$\frac{Q^S\mu(Q^S)f'(k^S)}{r+s+Q^S\mu(Q^S)}=r+s$$

implies that, denoting the capital labor ratio in the wage posting equilibrium by k^{wp},

$$k^{wp}=k^S$$

Therefore, with wage posting, capital investments are always efficient.

Why is this? You might think this is because there is no more holdup problem, and this is essentially true, but the intuition is a bit more subtle. In fact, there is something like hold-up because firms that invest more in equilibrium prefer to pay higher wages, but despite this the efficient level of investment results. The reason is that the higher wages that they pay is exactly offset with the higher probability that they will attract workers, so net returns are not subject to hold-up.

Next we have

$$\lambda = \frac{r + s + \eta(q)}{(r + s + \mu(q))q}$$

and substitute this into (iii), and used at zero profit constraints to solve for

$$w = f(k) - \frac{(r + s)(r + s + \eta(q))}{\eta(q)} k$$

Then we have:

$$\eta' \frac{q^2 f(k)}{r + s} + \left[r + s + \mu + \mu' q - q^2 \mu'\right] k = 0$$

which is identical to (13.2). We have therefore established:

Theorem 13.3. *The directed search equilibrium of the search and investment model is constrained efficient.*

Therefore, the equilibrium is constrained efficient! (note uniqueness is not guaranteed, but neither was it in the social optimum)

Thus, wage posting decentralizes the efficient allocation as the unique equilibrium.

How can we understand this efficiency better?

Acemoglu-Shimer consider a number of different economies

(1) Wage posting but no directed search. Clearly, in this case things are very bad, and we get the Diamond paradox.
(2) An economy where firms choose their own capital level, and then "post a bargaining parameter β" and upon matching, the firm and the worker Nash bargain with this parameter. It can be shown that if there is no capital choice, this economy will lead to an equilibrium in which all firms post the Hosios β, and constrained efficiency is achieved. But if there is a capital choice, and the only thing workers observe are the posted β's, then in equilibrium all firms offer the Hosios β, but there is under investment because of the hold-up problem.

(3) An economy where firms choose their own capital level and workers apply to firms observing these capital levels, and then they bargain according to some exogenously given parameter β. In this case, the equilibrium is inefficient and may have under or overinvestment. If the value of β is at the Hosios value, then the equilibrium will be constrained efficienct.

(4) An economy where firms choose their own capital level and post β, and workers observe both k and β, then always constrained efficiency.

So what do we learn? What is important is directed search, and especially the ability to direct search towards higher capital intensity firms. With wage posting, those are the high-wage firms, hence the objective is achieved. But the same outcome is also obtained if β is at the Hosios level, and workers observe capital levels.

Next, one might wonder whether an economy in which workers know/observe all of the wages offered in equilibrium is too extreme (especially given our motivation of doing away with a Walrasian auctioneer). A more plausible economy may be one where workers observe a finite number of wages.

Interestingly, we do not need all workers to observe all the wages as the model with a non-degenerate wage distribution in the last lecture illustrated.

THEOREM 13.4. *Suppose each worker observes (can apply to) at least two of the firms among the continuum of active firms, then the efficient allocation is an equilibrium of the search and investment model with directed search and wage posting.*

PROOF. (sketch) Suppose all firms are offering (q^{wp}, w^{wp}, k^{wp}). Now consider a deviation to some other (w', k'). Any worker who observes (w', k') has also observed another firm offering (w^{wp}, k^{wp}). Since (w^{wp}, k^{wp}) maximizes worker utility, he will apply to this in preference of

$$(w', k') \Longrightarrow q(w') = 0.$$

Consequently, all firms will be happy to offer (w^{wp}, k^{wp}) and they will each be tracked the queue length of q^{wp}. □

What is the intuition? *Effectively Bertrand Competition.* Each firm knows that it will effectively be competing with another firm offering the best possible deal to the worker, even though differently from the standard Bertrand model, it does not know which particular firm this will be. Nevertheless, the Bertrand reasoning forces each firm to go to the allocation that is best for the workers.

Note that this theorem is not stated as an "if and only if" theorem. In particular, when each worker only observes two wages, there can be other "non-efficient" equilibria. In particular, it can be proved that: *When each worker observes two wages, there can exist non-efficient equilibria.* This last theorem not withstanding, the conclusion of this analysis is that relatively little information is required for wage posting to decentralize the efficient allocation.

3. Risk Aversion in Search Equilibrium

The tools we developed so far can also be used to analyze general equilibrium search with risk aversion. Let us focus on the one-period model with wage posting. This can again be extended to the dynamic version, but explicit form solutions are possible only under constant absolute risk aversion (see Acemoglu-Shimer, JPE 1999)

Measure 1 workers; and they all have utility $u(c)$ where the consumption of individual i is

$$C_i = A_i + y_i - \tau_i$$

where A_i is the non-labor income of individual, y_i is his labor income, equal to the wage w that he applies it obtains if he's employed, and equal to the unemployment benefit z when unemployed. Finally, τ_i is equal to the taxes paid by this individual. u is increasing, concave and differentiable.

Let us start with a homogeneous economy where $A_i = A_0$ and $\tau_i = \tau$ for all i.

We also assume that firms are risk-neutral, which is not chill for example because workers may hold a balanced mutual fund. I will onlypresent the analysis for the static economy here.

Timing of events:

- Firms decide to enter, buy capital $k > 0$ (as before irreversible,) and post a wage w
- Workers observe all wage offers and decide which wage to seek (apply to).

As before, if on average there are q times as many workers seeking wage w as firms offering w, then workers get a job with prob. $\mu(q)$.

Firms fill their vacancies with prob. $\eta(q) \equiv q\mu(q)$, with our standard assumptions, $\mu'(q) < 0$ and $\eta'(q) > 0$

As before, let an allocation be $\langle \mathcal{W}, Q, K, U \rangle$, where $\mathcal{W}$ is the support of the wage distribution, $Q : \mathcal{W} \to \mathbb{R}$ is a queue length function, $K : \mathcal{W} \rightrightarrows \mathbb{R}$ is a capital choice correspondence, and $U \in \mathbb{R}$ is the equilibrium utility of unemployed workers.

DEFINITION 13.2. *An allocation is an equilibrium iff*

(1) $\forall w \in \mathcal{W}$ *and* $k \in K(w)$, $\eta(Q(w))(f(k) - w) - k = 0$.
(2) $\forall w, k$, $\eta(Q(w))(f(k) - w) - k \leq 0$.
(3) $U = \sup_{w \in \mathcal{W}} \mu(Q(w))u(A + w) + (1 - \mu(Q(w))u(A + z)$
(4) $Q(w)$ *s.t.* $\forall w$, $U \geq \mu(Q(w))u(A+w)+(1-\mu(Q(w)))u(A+z)$ *and* $Q(w) \geq 0$, *with complementary slackness.*

- $\Longrightarrow$ As before type of subgame perfection on beliefs about queue lengths after a deviation.

Characterization of equilibrium is similar to before

THEOREM 13.5. $(\mathcal{W}, Q, K, U)$ *an equilibrium if and only if* $\forall\ w^* \in \mathcal{W}$, $q^* \in Q(w^*), k^* \in K(w^*)$

$$(w^*,\ q^*, k^*) \in \arg\max \mu(q)u(A + w) + (1 - \mu(q))u(A + z)$$

s.t.

$$\eta(q)(f(k) - w) \geq 0.$$

In words, every equilibrium maximizes worker utility subject to zero profits, as proved before in the context of the risk-neutral model.

The analysis is similar to before. Profit maximization implies an even simpler condition (because the environment is static)

$$\eta(q^*)f'(k^*) = 1$$

Zero profits gives

$$\eta(q^*)(f(k^*) - w^*) = k^*$$

Now combining these two:

$$w^* = f(k^*) - k^* f'(k^*),$$

which you will notice is exactly the neoclassical wages equal to marginal product condition. Why is that?

Finally, combining this with, $\eta(q^*)f'(k^*) = 1$, we can derive a relation in the (q, w) space which corresponds to the zero-profits and profit maximization constraints that an equilibrium has to satisfy.

An equilibrium is then a tangency point between the indifference curves of homogeneous workers and this profit-maximization constraint, as we had in the risk-neutral model of Acemoglu-Shimer (IER, 1999):

The equilibrium can be depicted and analyzed diagrammatically.

Notice again that uniqueness **not** guaranteed.

What makes this attractive is that comparative statics can also be done in a simple way, exploiting "revealed preference" or single crossing.

For example, we have a change such that all workers become more risk-averse, i.e., and the utility function becomes more concave, what happens to equilibrium?

We can show that as risk-aversion increases, then we have $w \downarrow, q \downarrow, k \downarrow$.

Why? Indifference curves become everywhere steeper, the causing the tangency point to shift to the left. Unambiguous despite the fact that equilibrium may not be unique.

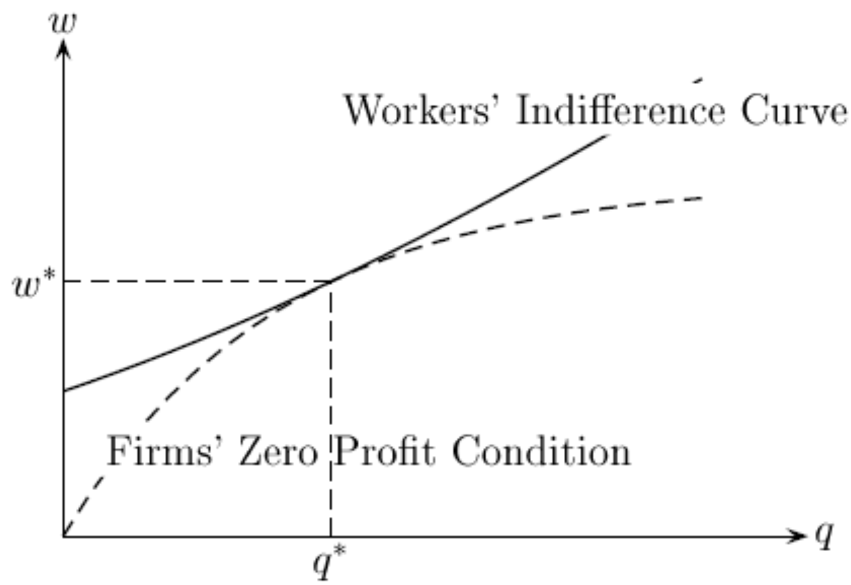

FIGURE 13.2

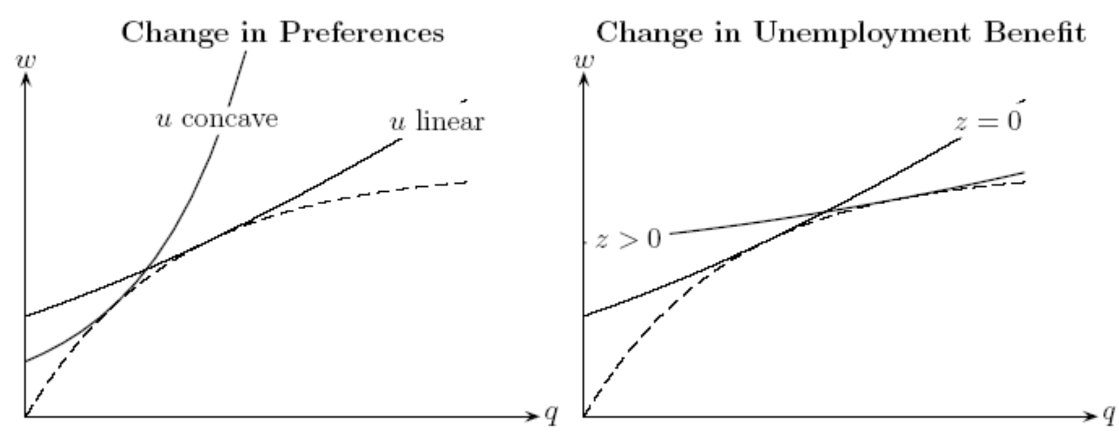

FIGURE 13.3

Essentially, comparative static result unambiguous because u_1-curve single-crosses u_2-curve.

Intuition: "Market Insurance." Workers are more risk-averse, so firms offer insurance by creating low-wage but easier to get jobs. Capital falls because once jobs are easier to get for workers, vacancies remain open for longer (with higher probability), so capital is unused for longer, reducing investment. Summarizing this:

THEOREM 13.6. *Consider a change from utility function u_1 to u_2 where u_2 is a strictly concave transformation of u_1. Then if (k_1, w_1, q_1) is any equilibrium with preferences u_1 and (k_2, w_2, q_2) is any equilibrium with preferences u_2, then $k_2 < k_1$, $w_2 < w_1$ and $q_2 < q_1$.*

Similarly, what happens when the unemployment benefits z increases from z_1 to z_2?

THEOREM 13.7. *Consider a change from unemployment benefits z_1 to $z_2 > z_1$. Then if (k_1, w_1, q_1) is any equilibrium with benefits z_1 and (k_2, w_2, q_2) is any equilibrium with benefits z_2, then $k_2 > k_1$, $w_2 > w_1$ and $q_2 > q_1$.*

PROOF. (sketch) By revealed preference

$$\begin{aligned} \mu(q_1)(u(A+w_1) - u(A+z_1)) &\geq \mu(q_2)(u(A+w_2) - u(A+z_1)) \\ \mu(q_2)(u(A+w_2) - u(A+z_2)) &\geq \mu(q_1)(u(A+w_1) - u(A+z_2) \end{aligned}$$

Multiply through and simplify

$$\begin{aligned} & (u(A+z_1) - u(A+z_2))(u(A+w_1) - u(A+w_2)) \geq 0 \\ \Longrightarrow \quad & z_1 \leq z_2 \Longleftrightarrow w_1 \leq w_2. \end{aligned}$$

All inequalities strict since all curves smooth. □

What happens when there is heterogeneity?

Suppose that there are $s = 1, 2, \ldots, S$ types of workers, where type s has utility function u_s, after-tax asset level A_s, and unemployment benefit z_s. Let U now be a vector in $\mathbb{R}^S$, and assume, for simplicity. Then:

THEOREM 13.8. *There always exists an equilibrium. If* $\{\mathcal{K}, \mathcal{W}, Q, U\}$ *is an equilibrium, then any* $k_s^* \in \mathcal{K}$,$w_s^* \in \mathcal{W}$, *and* $q_s^* = Q(w_s^*)$, *solves*

$$U_s = \max_{k,w,q} \mu(q) u_s(A_s + w) + (1 - \mu(q))\, u(A_s + z_s)$$

subject to $\eta(q)(f(k) - w) - k = 0$ *for some* $s = 1, 2, .., S$. *If* $\{k_s^*, w_s^*, q_s^*\}$ *solves the above program for some* s, *then there exists an equilibrium* $\{\mathcal{K}, \mathcal{W}, Q, U\}$ *such that* $k_s^* \in \mathcal{K}$, $w_s^* \in \mathcal{W}$, *and* $q_s^* = Q(w_s^*)$.

The important result here is that any triple $\{k_s^*, w_s^*, q_s^*\}$ that is part of an equilibrium maximizes the utility of one group of workers, subject to firms making zero profits. The market *endogenously* segments into S different submarkets, each catering to the preferences of one type of worker, and receiving applications only from that type.

The efficiency and output-maximization implications of this model are also interesting. First, supposed that $u(\cdot)$ is linear. Then $z = \tau = 0$ maximizes output. In particular, we have

THEOREM 13.9. *Suppose that* u *is linear, then* $z = \tau = 0$ *maximizes output.*

PROOF. (sketch) The equilibrium solves $\max \mu(q) w$ subject to $q\mu(q)(f(k) - w) = k$. Substituting for w we obtain:

$$\mu(q) f(k) - k/q \equiv y(k, q),$$

which is net output, thus is maximized by equilibrium choices. □

But an immediate corollary is that if $u(\cdot)$ is strictly concave, than the equilbrium with $z = \tau = 0$ does *not* maximize output.

THEOREM 13.10. *Suppose that* u *is strictly concave, then* $z = \tau = 0$ *does not maximize output.*

This is an immediate corollary of the previous theorems.

THEOREM 13.11. *Let* u *be an arbitrary concave utility function,* q^e *be the output-maximizing level of queue length and let*

$$z^e \equiv \frac{u(A_0 - \tau^e + w^e) - u(A_0 - \tau^e + z^e)}{u'(A_0 - \tau^e + w^e)}$$

and the balanced-budget condition

$$\tau^e = (1 - \mu(q^e))z^e$$

then the economy with unemployment benefit z^e *achieves an equilibrium with* q^e *and the maximum output.*

The following figure gives the intuition:

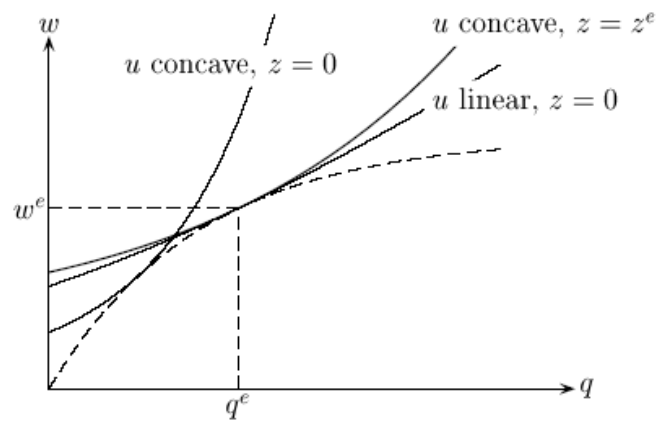

FIGURE 13.4

But this is not "optimal," since when workers are risk averse, maximizing output is not necessarily the right objective. Optimal unemployment benefits, z^o, should maximize ex ante utility. Interestingly, this could be greater or less than the efficient level of unemployment benefits, z^e, which maximizes output. What is the intuition for this?