# The Optimal Stopping Problem and the Fundamental Value Function in Search

[Sem0057]

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Academic Year 2024-2025

## Contents

# 1 The Optimal Stopping Problem and the Reservation Offer

## 1.1 DISCRETE TIME

Consider a risk neutral unemployed worker who is searching for a job under the following circumstances. Time is discrete. Each period the unemployed worker gets some income $b > 0$ and has a probability $\pi < 1$ of drawing an offer from some distribution of offers. Let the distribution of offers be indicated with $F(\tilde{W}) = prob\left\{W < \tilde{W}\right\}$ where $W$ is the lifetime utility associated with a job. Assume $F(0) = 0$ and $F(\tilde{W}^{MAX}) = 1$, with $\tilde{W}^{MAX} < \infty$. The distribution of offer is stationary and time invariant. Each period, conditional on getting an offer, the worker has to decide whether accepting or not the offer. Assume that if the worker accepts the offer, he never returns unemployed (a job for life!). How should the worker decide?. Should he accept any job, or is there some intrinsic value of waiting for another period? How can we determine this intrinsic value of waiting.

In formal terms, the worker has an option value at hand: *he or she has option to reject the offer and continue to search*. What we have to determine is when is optimal to stop searching and accept an offer. Seen in this term, the problem we want to solve is an optimal stopping time. We proceed very slowly, and we begin with the case in which time is discrete and finite, and we then move to the infinite horizon case.

### 1.1.1 FINITE CASE

Assume the worker has a discount rate equal to $\beta < 1$ and that he leaves for $T$ periods. Assume also that there is no possibility of recall. Consider an unemployed worker at time $t$ who has just received an offer which has values $W$. The worker faces two choices: accept the offer $W$ or continue to search and getting an offer next period. If the value of continue to search at time $t$ is $U_t$ (note that $U_t$ is to be determined!) the problem of the worker is simply solved by

$$(V_t|W) = \begin{cases} Max[U_t, W] & t = 1, 2, 3...t-1 \\ \\ W \quad \text{if} & t = T, \end{cases} \tag{1}$$

The maximization of the agent is whether to accept a $W$ job or continuing to search. Clearly, the worker should accept any job that pays at least $U_t$, so that it is optimal to stop searching if and only if $W \geq U_t$, where $U_t$ i*s the reservation offer*, or the value of the job at which the worker is indifferent between working or searching. We have learnt two important insights. First, $U_t$ can be interpreted as a value of a job that makes the worker indifferent between working or continuing. This maximization problem is described in Figure 1. Given $U_t$, the value function has a kinked exactly at $U_t$, which has indeed the interpretation of the reservation offer. Second, we understand that when the time is finite the problem is likely to be solved easily, since we know that in the last period of the game the worker will accept each and every job that has received. When time is over, the value of continue to search must be zero.

The worker has *the option to reject the offer and continue to search*, in which he or she receives $b$ next period in unemployment compensation, and may get a new offer $\tilde{W}$. Let $V_t = W$ the worker's lifetime utility when he has an offer that has value $W$.

The problem is to find $U_t$. To find $U_t$ let us imagine of considering a worker who has *just* rejected an offer and is continuing to search. At this point, it should be clear that the value of rejecting the offer and continue to search at time $t$ is

$$U_t = \begin{cases} \beta[b + (1-\pi)U_{t+1} + \pi \int \max[\tilde{W}, U_{t+1}] dF(\tilde{W})] & t = 1, 2, 3...t-1 \\ \\ 0 \quad \text{if} & t = T, \end{cases} \tag{2}$$

If $t = T$ the value of continuing is zero. If $t < T$ than the expression above becomes a proper value function. The right hand side is discounted at rate $\beta < 1$, since everything will happen next period. The next period income is $b$. Further, a new offer may or may not arrive. The offer arrives at rate $\pi$. If the offer does not arrive the worker will continue to search, and will get a value $U_{t+1}$. If an offer arrive how is the problem determined? It is described by the expression in the integral. But that is simply the problem described by equation 1 above in expected term, where we can get a value $\tilde{W}$ with probability $dF(\tilde{W})$. When will have $\tilde{W}$ on hand, we will do exactly what we did with equation 1 and Figure 1. The problem is now clear, the worker should in general reject any offer that does not pay at least $U_t$, which is indeed the value of the reservation offer.

$U_t$ then solves the discrete time Bellman equation. In the finite case, one can solves for the sequence $\{U_1, ...., U_T\}$ by backward induction. By working backward, the problem is fully determined. This is very easy to do it in a computer. A technical note: for $\beta < 1$ the right hand side is a contraction mapping by the Blackwell Sufficient conditions for a contraction (See Stokey Lucas).

### 1.1.2 INFINITE HORIZON WITH STATIONARY ENVIRONMENT

In the infinite horizon *stationary* case,

$$\lim_{t\to\infty} U_t = U$$

and the reservation value solves

$$U = \beta[b + (1-\pi)U + \pi \int \max[\tilde{W}, U]dF(\tilde{W})] \tag{3}$$

The value function $U$ can be further simplified by removing the max operator. This is a general property in reservation rule maximization. By virtue of the reservation property the integral in the previous expression can be written as

$$\int \max[\tilde{W}, U]dF(\tilde{W})] = \int_0^U UdF(\tilde{W}) + \int_U^{\tilde{W}^{Max}} \tilde{W}dF(\tilde{W})$$

Further, since $U$ can be written as $U = \int_0^U UdF(\tilde{W}) + \int_U^{\tilde{W}^{Max}} UdF(\tilde{W})$ the reservation rule of equation (3) becomes

$$U = \beta b + \beta U - \beta\pi \int_0^U UdF(\tilde{W}) - \beta\pi \int_U^{\tilde{W}^{Max}} UdF(\tilde{W}) + \beta\pi \int_0^U UdF(\tilde{W}) + \beta\pi \int_U^{\tilde{W}^{Max}} \tilde{W}dF(\tilde{W})$$

or

$$U = \frac{\beta}{1-\beta}[b + \pi \int_U^{\tilde{W}^{Max}} (\tilde{W} - U)dF(\tilde{W})] \tag{4}$$

which is an equation that implicitly defines $U$. Two comparative static follows.

**Remark 1.** *An increase in unemployment compensation $b$ increases the reservation offer, so that workers become more choosy*

To see this simply implicitly differentiate equation (4) with respect to $b$.

$$\begin{aligned}
\frac{\partial U}{\partial b} &= \frac{\beta}{1-\beta} + \frac{\beta\pi}{1-\beta}(U - U) - \frac{\beta\pi}{1-\beta}\frac{\partial U}{\partial b}\int_U^{\tilde{W}^{Max}} dF(\tilde{W}) \\
\frac{\partial U}{\partial b}[1 + \frac{\beta\pi(1-F(U))}{1-\beta}] &= \frac{\beta}{1-\beta} \\
\frac{\partial U}{\partial b} &= \frac{\beta}{1-\beta+\pi\beta(1-F(U))} \geq 0
\end{aligned}$$

**Remark 2.** *An increase in the arrival rate of offer $\pi$ increase the reservation offer, so that workers become more choosy*

Differentiate with respect to $\pi$ yields

$$\frac{\partial U}{\partial \pi}[1 + \beta F(U)] = \int_U^{\tilde{W}^{Max}} (\tilde{W} - U)dF(\tilde{W})$$

### 1.1.3 CONTINUOUS TIME

**POISSON PROCESSES** A Stochastic process $\{N(t), t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of "events" that have occurred up to time $t$.

A counting process is said to possess *independent increments* if the number of events which occur in disjoint time intervals are independent ($N(10)$ is independent of the event occurring between $N(15)$ and $N(10)$

A counting process is said to possess *stationary increments* if the distribution of the number of events which occur in any interval of time depends only on the length of the time interval

The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate $\lambda, \lambda > 0$ if *i)* $N(0) = 0$; *ii)* The process has stationary and independent increments; *iii)* $P\{N(h) = 1\} = \lambda h + o(h)$; *iv)* $P\{N(h) = 2\} = o(h)$.

Further, the number of events in any interval of length $t$ is Poisson distributed with mean $\lambda t$. That is for all $s, t \geq 0$

$$P\{N(t+s) - N(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad n = 0, 1, ..$$

which implies that $P\{N(t) = 0\} = e^{-\lambda t}$ (no events up to time $t$) has an exponential distribution with mean $1/\lambda$.

Let us denote the time of the first event by $T_1$. For $n > 1$, let $T_n$ denote the elapsed time between the $(n-1)st$ and $nth$ event. The sequence $\{T_n, n = 1, 2, ...\}$ is called the *sequence of interarrival times* Let's study the distribution of the $T_n$. Clearly, the event $\{T_1 > t\}$ takes place if and only if no events occur in the interval $[0, t]$ and thus,

$$P\{T_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

Hence $T_1$ has an exponential distribution with mean $1/\lambda$. Now $P\{T_2 > t\} = E[P\{T_2 > t|T_1\}]$. However

$$\begin{aligned}[P\{T_2 &> t|T_1 = s\} = P\{0 \text{ events in } (s, s+t)|T_1 = s\} \\ &= P\{0 \text{ events in } (s, s+t)\} \\ &= e^{-\lambda t}\end{aligned}$$

where the last two equations followed from independent and stationary increments. This implies that $T_2$ is also exponential with mean $1/\lambda$. Thus, $T_n, n = 1, 2.....$ *are independent identically distributed exponential random variables having mean* $1/\lambda$. This implies that the assumption of stationary independent increments is equivalent to asserting that, at any point in time, the process probabilistically restarts itself. In other words, the *process has no memory.*

**THE ASSET VALUE FUNCTION** Let's assume that the interval between time $t$ and time $t+1$ is of length $\Delta$ and let assume that the arrival rate of job offer follows a Poisson process with arrival rate $\lambda$. Equation (1) can be rewritten as

$$U(t) = \frac{1}{1+r\Delta}[b\Delta + (1 - \lambda\Delta - o(\Delta))U(t+\Delta) + (\lambda\Delta + o(\Delta))\int \max[\tilde{W}, U(t+\Delta)]dF(\tilde{W})] \tag{5}$$

where $\beta = \frac{1}{1+r\Delta}$. Equation 5 can be written as

$$U(t)r\Delta - [U(t+\Delta) - U(t)] = b\Delta - (\lambda\Delta + o(\Delta))U(t+\Delta) + (\lambda\Delta + o(\Delta))\int \max[\tilde{W}, U(t+\Delta)]dF(\tilde{W})]$$

and diving by $\Delta$ yields

$$rU(t) - \frac{[U(t+\Delta) - U(t)]}{\Delta} = b - (\lambda + \frac{o(\Delta)}{\Delta})U(t+\Delta) + (\lambda + \frac{o(\Delta)}{\Delta})\int \max[\tilde{W}, U(t+\Delta)]dF(\tilde{W})]$$

and taking the limit as $\Delta \to 0$

$$rU(t) = b + \lambda[\int \max[\tilde{W}, U(t)]dF(\tilde{W})] - U(t)] + \dot{U}(t) \tag{6}$$

since $\frac{o(\Delta)}{\Delta} \to 0$ as $\Delta \to 0$ and

$$\dot{U}(t) = \lim_{\Delta \to 0} \frac{[U(t+\Delta) - U(t)]}{\Delta}.$$

In equation (2) $U(t)$ is the "asset" or "option" value of search activity. In this interpretation, Equation (2) prices the option by requiring that the opportunity cost of holding it (the left hand side) be equal to the current income flow, plus the expected capital gain flows (the product of the arrival frequency $\lambda$ and the expected capital gain given an offer arrival), and a pure rate of capital gain or loss attributable to waiting another instant for an offer arrival. The value of an optimal search strategy must also satisfy the transversality condition

$$\lim_{t \to \infty} U(t)e^{-rt} = 0$$

In the stationary case $U(t) = U$ and the asset value function reads

$$rU = b + \lambda[\int \max[\tilde{W}, U)]dF(\tilde{W})] - U].$$

Again, the reservation value function can be written without max operator as

$$rU = b + \lambda[\int_U^{W^{MAX}} [\tilde{W} - U]dF(\tilde{W})$$

.

If $\tilde{W}$ is degenerate and can take only 1 possible value $W$ and if the labor market is *viable* ($W > U$) the asset function reads

$$rU = b + \lambda[W - U]$$

which is the simplest value function we will be playing with, and the one that is used in the baseline matching model (see Handout #2).

### 1.1.4 Solving the time-varying value function

This is useful for solving and integrating forward time varying value functions.

$$(\delta + r)J(t) = p(t) - w(t) + \frac{d}{dt}J(t) \tag{7}$$

Rewrite it as

$$\dot{J} - (\delta + r)J(t) = -p(t) + w(t)$$

where $\dot{J} = \frac{d}{dt}J(t)$. Multiplying both sides by $e^{-(\delta+r)t}$ yields

$$e^{-(\delta+r)t}[\dot{J} - (\delta + r)J(t)] = e^{-(\delta+r)t}[-p(t) + w(t)] \tag{8}$$

Noting that in the LHS

$$e^{-(\delta+r)t}[\dot{J} - (\delta + r)J(t)] = \frac{\partial}{\partial t}[e^{-(\delta+r)t}J(t)]$$

we can integrate from to $t$ up to infinite both sides of equation 8to yield

$$\int_t^{\infty} \frac{\partial}{\partial s} e^{-(\delta+r)s}J(s)ds = -\int_t^{\infty} e^{-(\delta+r)s}[p(s) - w(s)]ds$$

$$\left| e^{-(\delta+r)s}J(s) \right|_t^{\lim s\to\infty} = -\int_t^{\infty} e^{-(\delta+r)s}[p(s) - w(s)]ds$$

or simply

$$\lim_{s\to\infty} e^{-(\delta+r)s}J(s) - J(t)e^{-(\delta+r)t} = -\int_t^{\infty} e^{-(\delta+r)s}[p(s) - w(s)]ds$$

Using the transversality condition

$$\lim_{s\to\infty} e^{-rs}J(s) = 0$$

the value function reads

$$J(t) = e^{-(\delta+r)t}\int_t^{\infty} e^{-(\delta+r)s}[p(s) - w(s)]ds$$

or

$$J(t) = \int_t^{\infty} e^{-(\delta+r)(s-t)}[p(s) - w(s)]ds$$