Chapter 1
The Benchmark Model: A Single Worker Firm

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Outline

1. Matching and separation in the labor market
2. The job creation condition
3. Wages
4. Equilibrium
5. Further discussion of wages
6. Endogenous job destruction

Matching and separation in the labor market

Matching in the labor market

- At a given point in time t, a number of vacant positions $\mathcal{V}_t$ coexist with a number of unemployed job seekers $\mathcal{U}_t$ looking for jobs in the labor market.
- The labor force is normalized to $\mathcal{N}_t + \mathcal{U}_t = 1$, where $\mathcal{N}_t$ is the number of employed workers.

Consider a period of time, which may be a discrete time period or alternatively an infinitely small time interval in continuous time.

- Let f_t be the per-period rate at which unemployed workers find a job.
- Let also q_t be the per-period rate at which vacancies are filled.

An identity in the labor market is that the total number of matches formed per unit of time $\mathcal{M}_{Lt}$ is both the total number of filled vacancies and the number of hired workers:

$$\mathcal{M}_{Lt} = q_t \mathcal{V}_t \equiv f_t \mathcal{U}_t \tag{1}$$

Matching in the labor market

In discrete time, since q_t and f_t are probabilities in the $(0, 1)$ interval, we necessarily have that

$$\mathcal{M}_{Lt} \leq \min(\mathcal{U}_t, \mathcal{V}_t) \tag{2}$$

- we will assume this to be the case throughout.
- In continuous time, the values of q_t and f_t are not bounded.
- However, in an arbitrarily small time interval dt, matching probabilities $f_t dt$ and $q_t dt$ are also below unity as dt approaches zero.
- We typically have in mind a time period of a week or a month, sometimes even a quarter.

Matching in the labor market

Labor market tightness : a measure of labor market conditions

$$\theta_t = \frac{\mathcal{V}_t}{\mathcal{U}_t} \tag{3}$$

- A tight labor market is one in which job seekers are scarce relative to the availability of jobs.
- It naturally follows from equation (**??**) that f_t and q_t are necessarily linked through θ_t, even in the absence of structural assumptions on the matching process:

$$f_t \equiv \theta_t q_t \tag{4}$$

Equation (**??**) implies that f_t and q_t cannot be simultaneously exogenous to labor market conditions summarized in θ_t.

Matching in the labor market

The tradition of the search literature is to assume a constant returns to scale matching function:

$$\mathcal{M}_{Lt} = \mathcal{M}_L(\mathcal{V}_t, \mathcal{U}_t) \quad (5)$$
$$\text{with } \partial \log \mathcal{M}_L / \partial \log \mathcal{U}_t = \eta_L(\theta_t)$$
$$\text{and } \partial \log \mathcal{M}_L / \partial \log \mathcal{V}_t = 1 - \eta_L(\theta_t)$$

- $\eta_L(\theta_t)$ is the elasticity of matching with respect to the amount of job seekers.
- In general, this elasticity may depend on the current tightness of the labor market.

Matching in the labor market

Summarizing:

- vacant jobs and unemployed workers match at a rate determined through a constant returns to scale matching function $\mathcal{M}_L(\mathcal{V},\mathcal{U})$,
- the transitionrates for vacancies and for unemployed workers, respectively, are given by:

$$\frac{\mathcal{M}_L(\mathcal{V}_t,\mathcal{U}_t)}{\mathcal{V}_t} = q(\theta_t) \quad \text{with } q'(\theta_t) \leq 0 \tag{6}$$

$$\frac{\mathcal{M}_L(\mathcal{V}_t,\mathcal{U}_t)}{\mathcal{U}_t} = \theta_t q(\theta_t) = f(\theta_t) \quad \text{with } f'(\theta_t) \geq 0 \tag{7}$$

Matching in the labor market

Under iso-elastic functions f_t and q_t of θ_t, one would recover a Cobb-Douglas matching function, $\mathcal{M}_{Lt} = \chi_L \mathcal{U}_t^{\eta_L} \mathcal{V}_t^{1-\eta_L}$ with $\chi_L > 0$ and $\eta_L \in (0, 1)$.

- In this instance the job filling and job finding rates are, respectively, $q(\theta_t) = \chi_L \theta_t^{-\eta_L}$ and $f(\theta_t) = \chi_L \theta_t^{1-\eta_L}$.
- A feature of this function is a constant elasticity of matching with respect to unemployment $\eta_L(\theta_t) = \eta_L$,

Matching in the labor market

In discrete time settings, a common alternative is the functional form $\mathcal{M}_{Lt} = \mathcal{V}_t\mathcal{U}_t / \left(\mathcal{V}_t^{\nu_L} + \mathcal{U}_t^{\nu_L}\right)^{1/\nu_L}$, where $\nu_L > 0$.

- This function, proposed by DenHannRameyWatsonAER2000 has the appealing feature of being bounded between 0 and 1.
- The job filling and job finding rates are, respectively, $q(\theta_t) = \left(1 + \theta_t^{\nu_L}\right)^{-1/\nu_L}$ and $f(\theta_t) = \left(1 + \theta_t^{-\nu_L}\right)^{-1/\nu_L}$.
- The elasticity of matching with respect to job seekers in this instance depends on the current level of labor market tightness, $\eta_L(\theta_t) = \theta_t^{\nu_L}\left(1 + \theta_t^{\nu_L}\right)^{-1}$.

Separation

Once formed, matches create profit streams, and one needs to specify the separation process.
We assume that profit streams end at exogenous rate s. This generic notation hides a number of possible events, including:

(i) as the firm's bankruptcy, whatever the reason, such as a financial or credit shock, arriving at rate s^C;

(ii) as turnover of the worker leading to the job becoming vacant, an event occurring at rate s^L; or

(iii) a lack of demand due to changes in consumer tastes, an event that would occur at rate s^G.

Once financial and goods market frictions are introduced, the distinction will be meaningful and we need to separate these three different concepts of profit termination. Here they are confounded.

The job creation condition

The job creation condition - Continuous time

$$rJ_\pi = x - w + s\left(J_v - J_\pi\right) + \partial J_\pi / \partial t \tag{8}$$

$$rJ_v = -\gamma + q(\theta)\left(J_\pi - J_v\right) + \partial J_v / \partial t \tag{9}$$

- Firms create vacancies by paying posting costs $\gamma > 0$ per unit of time the job is open and unfilled.
- Once matched with a worker, they produce a flow of output x and pay a wage w to the worker, resulting in flow profits $x - w$.
- r denotes the rate of interest for discounting time.
- J_π and J_v are the value functions of the filled position and the vacant position.

The job creation condition - Continuous time

$$rW_n = w + s\,(W_u - W_n) + \partial W_n/\partial t \tag{10}$$

$$rW_u = z + \theta q(\theta)\,(W_n - W_u) + \partial W_u/\partial t \tag{11}$$

- W_n and W_u are the value functions of the employed worker and the unemployed worker.
- z the value of non-employment for the worker. It is a composite of unemployment benefits and other income support programs for the unemployed, and the value of leisure and other non-market activities such as home production to the worker.

A steady-state equilibrium implies that all partial time derivatives are equal to zero.

The job creation condition

Free Entry: It is assumed that firms may freely enter the labor market.

- This implies that the value of a vacancy is identically equal to zero in equilibrium, a result referred to as the free entry condition:

$$J_v \equiv 0 \text{ for all } t \tag{12}$$

From equation (**??**) and the value functions (**??**) and (**??**) above, we derive an equilibrium job creation condition:
Continuous time job creation condition:

$$\frac{\gamma}{q(\theta)} = \frac{x - w}{r + s} \tag{13}$$

The job creation condition

Equilibrium: present discounted costs = present discounted profits

- *cost* of a vacancy at the time of entry:
  $\gamma / [r + q(\theta)] = Q_v(\theta) \times \gamma / q$
- value of *profits* at the time of entry: $Q_v(\theta) \times (x - w)/(r + s)$

When deciding to enter the market, tomorrow's profit flows have to be multiplied by a discounting coefficient $q/(r+q)$. This discounting reflects how much time it will take to match with a worker.

- $Q_v(\theta) \equiv q[\theta] / [r + q(\theta)]$: the value of a particular asset that would bring one dollar of income after a random delay, with the event leading to that dollar occurring with probability q per unit of time.
  - Under no discounting, that value is 1.
  - Under a strictly positive discount rate, it has a value $q/(r+q) < 1$.
  - When matching is infinitely fast, $q \to \infty$ and the ratio $\to$ to 1.
  - When matching is infinitely slow, the ratio goes to zero.

Existence and uniqueness

Existence of a solution is ensured by the condition

$$\frac{x - w(0)}{r + s} \geq \frac{\gamma}{q(0)} = 0$$

- the lowest possible value of labor market tightness, $\theta = 0$, for which a firm can enter and make more profit than the expected costs of entering the market.

For a given wage, given the assumed monotonicity of q in θ, there is at most one value of labor market tightness satisfying equation (**??**).

- The same remains true for all wage functions if they are increasing in labor market tightness
- the right-hand side of (**??**) is either constant (if the wage is constant) or decreasing in θ (if the wage increases in θ), while search costs in the left-hand side are increasing in θ.

Existence and uniqueness

The value of a vacancy, after replacing the value of a filled position J_π from equation (**??**) into the value of a job vacancy, can be expressed as a function of θ:

$$rJ_v(\theta) = \frac{r+s}{r+s+q(\theta)}(-\gamma) + \frac{q(\theta)}{r+s+q(\theta)}(x-w) \qquad (14)$$

- the value of a vacancy out of equilibrium appears as a weighted average of flow recruiting costs (negative term) and of future profit streams (strictly positive term under the assumption that $x > w$).
- the weight $q(\theta)/(r+s+q(\theta))$ varies monotonically between 0 and 1 as θ describes the full range from 0 to infinity.
- there is a unique value of θ for which $rJ_v(\theta)$ crosses the horizontal axis at zero. Further, any deviation implies a return to the equilibrium value:
  - if θ is above the equilibrium value, the value of a vacancy is negative, leading vacancies to exit the labor market and reducing the number of vacancies $\mathcal{V}$. This leads to a decrease in labor market tightness at a fixed unemployment rate $\mathcal{U}$.

Beveridge curve

A second important building block of the model is the steady-state condition for the stock of unemployment.
Unemployment evolves according to

$$\frac{d\mathcal{U}}{dt} = s(1-\mathcal{U}) - f(\theta)\mathcal{U} \tag{15}$$

- $s(1-\mathcal{U})$: flows out of employment into unemployment,
- $f(\theta)\mathcal{U}$ are flows out of unemployment into employment.

A steady state $d\mathcal{U}/dt = 0$ requires the equality of the flows in and out of unemployment, leading to a steady-state rate of unemployment as a function of the transition rates:

$$\mathcal{U} = \frac{s}{s + f(\theta)} \tag{16}$$

This downward sloping relation between vacancies and unemployment, which matches the empirical observation of a negative correlation between unemployment and vacancies, the so-called Beveridge curve.

Discrete time job creation condition

Begin from the perspective of a firm

- choosing the number of job vacancies to post at unit cost γ,
- knowing and taking as given the probability of meeting a worker $q(\theta_t)$

The firm also takes as given the discrete time law of motion for employment, adapted from equation (**??**). Using the fact that at any time, $\mathcal{U}_t + \mathcal{N}_t = 1$, we have

$$\mathcal{N}_{t+1} = (1 - s)\mathcal{N}_t + q(\theta_t)\mathcal{V}_t \qquad (17)$$

Discrete time job creation condition

this firm's problem equates the cost of posting γ to the expected discounted value of a hired worker $J_{\pi t+1}$, conditional on meeting a job seeker, through the probability $q(\theta_t)$, and on information at time t, through the expectations operator $\mathbb{E}_t$.

$$\gamma = q(\theta_t)\beta\mathbb{E}_t[J_{\pi t+1}] \tag{18}$$

The value of a hired worker to the firm, given by

$$J_{\pi t} = x_t - w_t + \beta(1-s)\mathbb{E}_t[J_{\pi t+1}] \tag{19}$$

is the sum of the per period profit flow $x_t - w_t$ and of the discounted value of an additional period of productive employment, conditional on survival to the next period, which occurs with probability $(1-s)$.

Discrete time job creation condition

Discrete time job creation condition:

$$\frac{\gamma}{q(\theta_t)} = \beta \mathbb{E}_t \left[x_{t+1} - w_{t+1} + (1-s)\frac{\gamma}{q(\theta_{t+1})} \right] \qquad (20)$$

- The left-hand side of (**??**) can be interpreted as the average cost of filling a job vacancy, inversely related to the meeting rate $q(\theta_t)$.
- The right-hand side is the discounted expected value to the firm of a filled job vacancy, conditional on the state of the economy at date t. This is comprised of a period profit flow $(x_{t+1} - w_{t+1})$ and a continuation value should the employment relationship survive to the next period.

Discrete time job creation condition

At a steady state this job creation condition becomes

$$\frac{\gamma}{q(\theta)} = \frac{x - w}{r + s} \tag{21}$$

- This is identical to the continuous time steady state condition in equation (**??**).
- There is thus an equivalence between the two assumptions on time, continuous or discrete, at a steady state.

The discrete time law of motion for unemployment,

$$\mathcal{U}_{t+1} = s\,(1 - \mathcal{U}_t) + [1 - f(\theta_t)]\mathcal{U}_t$$

also leads to a steady-state unemployment equation identical to (**??**), except that s and f are transition probabilities and not transition intensities.

Wages

Surplus and reservation wages

Matches create economic rents for workers and firms, which are measured as the private surplus of the relationship.

In general, both firm and worker are willing to maintain a relationship as long as the private surplus is positive.

Define the total surplus to a match by

$$\Sigma_l^T = \Sigma^f(w) + \Sigma^n(w)$$

- $\Sigma^f(w) = J_\pi(w) - J_v$: the labor surplus of the firm
  - difference between the value of an employed worker paid a wage w, $J_\pi(w)$, and of searching in the labor market, J_v
- $\Sigma^w(w) = W_n(w) - W_u$: the labor surplus of the worker
  - difference between the value of working for a wage w, $W_n(w)$, and the value of unemployment, W_u

Surplus and reservation wages

We also define the reservation wages of both the worker and the firm.

- $\underline{w}_n$: The *lowest* wage a worker would be willing to accept
- $\underline{w}_f$: The *highest* wage a firm would be willing to pay

They are such that $\Sigma^f(\underline{w}_f) = 0$ and $\Sigma^n(\underline{w}_n) = 0$.

Surplus and reservation wages

In continuous time, the asset values of unemployment and employment are defined by

$$rW_u = z + f(\theta)(W_n - W_u) + \partial W_u / \partial t \quad (22)$$

$$rW_n = w + s(W_u - W_n) + \partial W_n / \partial t \quad (23)$$

In discrete time, we have:

$$W_{ut} = z + \beta \mathbb{E}_t \left[f(\theta_t) W_{nt+1} + (1 - f(\theta_t)) W_{ut+1} \right] \quad (24)$$

$$W_{nt} = w_t + \beta \mathbb{E}_t \left[s W_{ut+1} + (1 - s) W_{nt+1} \right] \quad (25)$$

Surplus sharing and Nash bargaining over wages

The Nash solution has become the most popular in macroeconomics, following Mortensen's (1986) handbook survey.

- This wage allocates a share of the total surplus Σ_I^T to each party in the bargaining game.
- A corollary is that the joint surplus Σ_I^T needs to be strictly positive for a match to survive.

This approach assumes that the wage is negotiated every period or, in continuous time, at any time if the outcome of negotiation leads to a different wage, as is the case if underlying parameters evolve over time.

This convenient, yet strong, assumption can be relaxed as discussed later on.

Surplus sharing and Nash bargaining over wages

The Nash wage solves the general maximization problem:

$$w_t = \text{argmax}\left[W_{nt}(w_t) - W_{ut}\right]^{\alpha_L}\left[J_{\pi t}(w_t) - J_{vt}\right]^{1-\alpha_L} \tag{26}$$

- $\alpha_L \in (0, 1)$ denotes the relative bargaining weight of the worker in wage setting.

From the Bellman equations for J_π and W_n above (either in continuous or in discrete time),

- the worker's surplus is linearly increasing in the wage
- while the firm's surplus is linearly decreasing in the wage, with opposite slopes.

Thus the Nash problem's objective function of equation (**??**) is increasing and concave.

- It is equal to zero at the worker's reservation wage $\underline{w}_n$, reaches a maximum, and then decreases once again to zero at the firm's reservation wage $\underline{w}_f$.

Surplus sharing and Nash bargaining over wages

The solution to this problem is a wage that results in a surplus-sharing rule:

$$(1-\alpha_L)\,(W_{nt}-W_{ut}) = \alpha_L\,(J_{\pi t}-J_{vt}) \tag{27}$$

conveniently re-expressed as either

- Worker's surplus $(W_{nt}-W_{ut}) = \alpha_L \Sigma_{lt}^T$, or
- Firm' surplus $(J_{\pi t}-J_{vt}) = (1-\alpha_L)\,\Sigma_{lt}^T$.

The latter expressions focus on the Nash wage as dividing the total surplus of the match between worker and a firm into shares α_L and $(1-\alpha_L)$, respectively.

Surplus sharing and Nash bargaining over wages

In continuous time, using equations (**??**) and (**??**), the Nash-wage can readily be expressed as the weighted average of the marginal product of labor and the value of the worker's outside option, here unemployment:

$$w = \alpha_L x + (1 - \alpha_L) r W_u \tag{28}$$

This illustrates the main forces affecting the Nash wage.

- It is increasing in the marginal product of labor, with the worker receiving a share α_L of the increase.
- It is also increasing in the value of being unemployed and searching in the labor market for another employer, itself a function of the rate at which workers find jobs, $f(\theta)$.

Note that this expression is true both in the steady-state and when the value function varies with time, to the extent that bargaining takes place in continuous time too.

Surplus sharing and Nash bargaining over wages

Expanding this expression leads to the wage rule in equation (**??**) in continuous time, and (**??**) in discrete time.

Nash wage:

$$\text{Continuous time:} \quad w = \alpha_L \left(x + \gamma\theta\right) + (1 - \alpha_L)z \tag{29}$$

$$\text{Discrete time:} \quad w_t = \alpha_L \left(x_t + \gamma\theta_t\right) + (1 - \alpha_L)z \tag{30}$$

Surplus sharing and Nash bargaining over wages

Two elements of this wage originate in the worker's outside option rW_u in equation (**??**).

1. the wage is increasing in the flow value of unemployment, z
2. the wage is increasing in labor market tightness θ
   - This captures the cost of recruiting for firms which pay a flow cost γ for open job vacancies as labor market tightness determines the frequency of meeting workers.

The Nash-solution can be extended to concave utility functions as long as agents cannot smooth consumption:

- Under concave utility $v(w)$, the Nash-maximand leads to different effective shares $\tilde{\alpha}_L(v') = \alpha_L \dfrac{v'}{1 - \alpha_L(1 - v')}$, increasing in marginal utility v'.

Equilibrium

Equilibrium

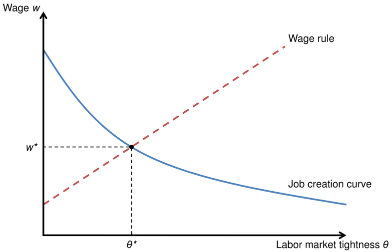
- A steady-state equilibrium is a pair (θ^*, w^*) that satisfies the job creation condition in (**??**) and wage rule in (**??**) or (**??**).
- Combining the wage rule and job creation condition, equilibrium labor market tightness is determined by:

Equilibrium job creation condition:

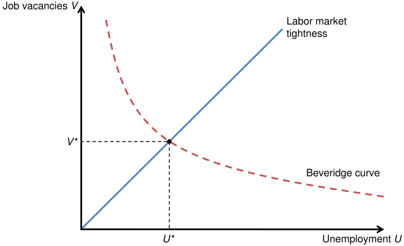
$$\frac{\gamma}{q(\theta^*)} = \frac{(1-\alpha_L)(x-z) - \alpha_L \gamma \theta^*}{r+s} \tag{31}$$

- As long as $x > z$, a steady-state equilibrium exists and is unique.

Equilibrium



(a) Equilibrium labor market tightness and wage



(b) Equilibrium unemployment and vacancies

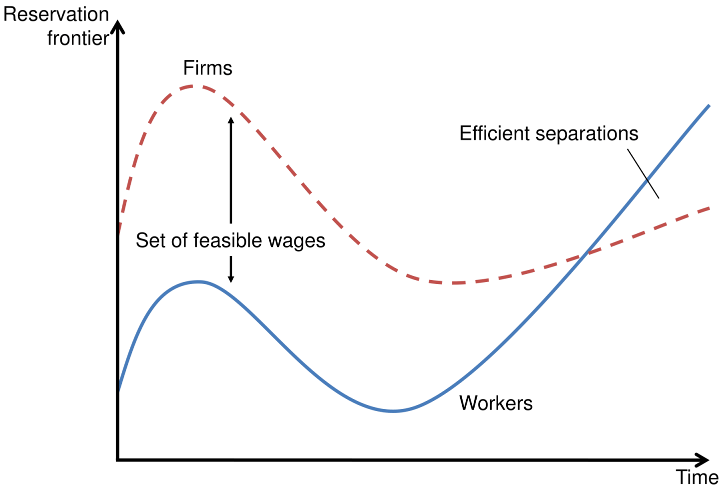
Figure: Steady-state equilibrium θ^*, w^*, $\mathcal{U}^*$, and $\mathcal{V}^*$ in the benchmark model

Further discussion of wages

Acceptance Set of Wages

- Wages must be above the minimal acceptable wage for workers $\underline{w}_n$
- Wages must be below the maximal acceptable wage of firms $\underline{w}_f$
- Viability condition for a job: $\underline{w}_n \leq \underline{w}_f$
- Any wage in the set $[\underline{w}_n, \underline{w}_f]$ could be a feasible wage
- An efficient separation of reservation wages arises when the value of a worker's reservation wage is above the value of a firm's reservation wage

Example of a feasible wage set over time and separations



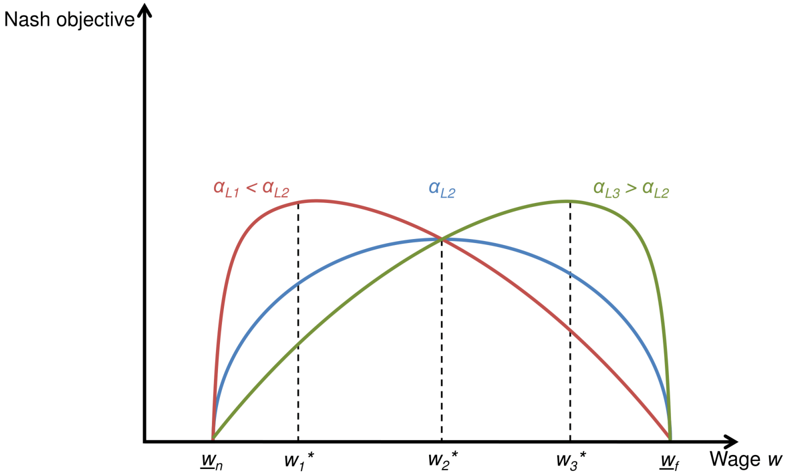
Acceptance Set of Wages

- Calculation of reservation wages in continuous time:

$$W_n(\underline{w}_n) = W_u \Leftrightarrow \underline{w}_n = z + [f(\theta) + s]\left(W_n' - W_u\right) - \partial(W_n - W_u)/\partial t J_n \quad (32)$$

- In discrete time:
- A reinterpretation of the Nash program uses the reservation wages

Nash Bargaining Representation with Nash Maximand



Credible Bargaining

- The indifference condition for a worker when considering a wage offer is:

$$W_{nt}^{w} = \varphi W_{ut} + (1-\varphi)z + \frac{1-\varphi}{1+r'}\mathbb{E}_t W_{nt+1/M}^{w'} \qquad (33)$$

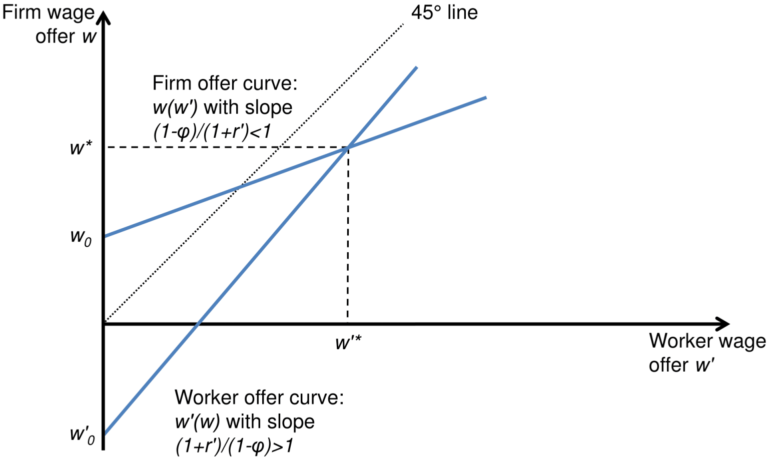
- The indifference condition for the firm when considering the worker's offer is:

$$J_{\pi t}^{w'} = \varphi J_{vt} + (1-\varphi)\left(-\zeta + \mathbb{E}_t\left[\frac{1}{1+r'}J_{\pi t+1/M}^{w}\right]\right) \qquad (34)$$

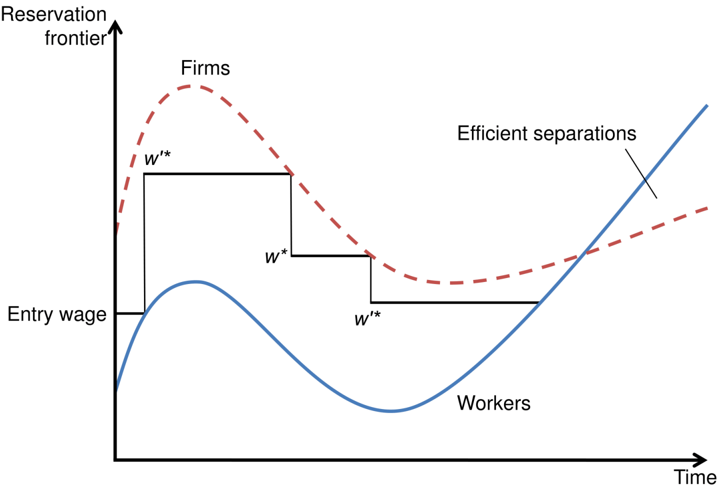
$$\text{Firm offer curve: } w(w') \Rightarrow w = \bar{w}_0 + \frac{1-\varphi}{1+r'}w'$$

$$\text{Worker offer curve: } w'(w) \Rightarrow w = \bar{w}'_0 + \left(\frac{1-\varphi}{1+r'}\right)^{-1} w'$$

Credible Bargaining Representation



Combining MacLeod-Malcomson



Endogenous job destruction

Endogenous Job Destruction

- Assume that the match between worker and firm draws productivity a from distribution $G(a)$
- Assume the entry level of a is random.
- The Bellman Equations for worker and firm are now:

$$rW_u = z + \theta q(\theta) \left\{ \int \max \left[W_n(a'), W_u \right] \mathrm{d}G(a') - W_u \right\} \tag{35}$$

$$rW_n(a) = w(a) + s[W_u - W_n(a)] + \mu \left\{ \int \max \left[W_n(a'), W_u \right] \mathrm{d}G(a') - W_n(a) \right\} \tag{36}$$

$$rJ_v = -\gamma + q(\theta) \left\{ \int \max \left[J_\pi(a'), J_v \right] \mathrm{d}G(a') - J_v \right\} \tag{37}$$

$$rJ_\pi(a) = ax - w(a) + s(J_v - J_\pi(a)) + \mu \left\{ \int \max \left[J_\pi(a'), J_v \right] \mathrm{d}G(a') - J_\pi(a) \right\} \tag{38}$$

Endogenous Job Destruction

- The economic surplus of a match with productivity a, summing individual surpluses from the above Bellman equation is:

$$(r+s+\mu)\,\Sigma_l^T(a) = ax+\mu\int \max\left[\Sigma_l^T(a'),0\right]\mathrm{d}G(a')-rW_u-rJ_v$$

- Job destruction condition:

$$rW_u + rJ_v \quad = Ax+ \quad \mu\int_A \Sigma_l^T(a')\mathrm{d}G(a')$$

Endogenous Job Destruction

- Inserting a solution $\Sigma_I^T(a) = \dfrac{(a-A)x}{r+s+\mu}$, along with the value of the worker's outside option when wages are set by Nash bargaining, leads to a job destruction condition in equation
- There exists a unique equilibrium pair (θ, A) that satisfies the job destruction condition and a new job creation condition:

  Continuous time job destruction condition:

$$z + \frac{\alpha_L}{1-\alpha_L}\gamma\theta = Ax + \frac{\mu x}{r+s+\mu}\int_A \left(a' - A\right) \mathrm{d}G(a') \quad (39)$$

  Continuous time job creation condition:

$$\frac{\gamma}{q(\theta)} = \frac{(1-\alpha_L)\, x}{r+s+\mu}\int_A \left(a' - A\right) \mathrm{d}G(a') \quad (40)$$

Endogenous Job Destruction

- Employed workers now separate from the firm at an endogenous rate $G(A)$, and an exogenous rate s. The total separation rate is:

$$s^T(A) = s + \mu G(A)$$

- Unemployment now evolves according to:

$$\frac{d\mathcal{U}}{dt} = [s + \mu G(A)](1 - \mathcal{U}) + [1 - G(A)] f(\theta)\mathcal{U}$$

- As a result, steady state unemployment is now given by

$$\mathcal{U} = \frac{s^T(A)}{s^T(A) + [1 - G(A)] f(\theta)}$$