The Optimal Stopping Problem and the Fundamental Value Function in Search

[Sem0057]

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Contents

1 The Optimal Stopping Problem and the Reservation Offer

1.1 DISCRETE TIME

- Consider a risk neutral unemployed worker who is searching for a job under the following circumstances.
- Time is discrete.
- The worker is impatient and discount the future at rate $\beta < 1$
- Each period the unemployed worker gets some income $b > 0$
- The individual has a probability $\pi < 1$ of drawing an offer from some distribution of offers.
- A jobs is worth W, and think at W as the PDV of a job that last forver.
- Where does W come from?
- Let the distribution of offers be indicated with $F(\tilde{W}) = prob\left\{W < \tilde{W}\right\}$ where W is the lifetime utility associated with a job.
- F is exogenous
- Assume $F(0) = 0$ and $F(\tilde{W}^{MAX}) = 1$, with $\tilde{W}^{MAX} < \infty$.
- The distribution of offer is stationary and time invariant. Each period, conditional on getting an offer, the worker has to decide whether accepting or not the offer. Assume that if the worker accepts the offer, he never returns unemployed (a job for life!).

- **What is the Economics of this problem?** What is the choice of the worker?
- We consider a unemployed worker that has an offer on hand worth W
  1. ACCEPT. She gets W forever
  2. REJECT. She gets unemployed income plue expected value of future offers
- Seen in this way, rejecting meens keep searching fro better jobs
- WHich jobs should hte unemployed accept ?
- How should the worker decide?. Should he accept any job, or is there some intrinsic value of waiting for another period? How can we determine this intrinsic value of waiting.
- In formal terms, the worker has an option value at hand: *he or she has option to reject the offer and continue to search*. What we have to determine is when is optimal to stop searching and accept an offer.
- Seen in this term, the problem we want to solve is an optimal stopping time.
- We proceed very slowly, and we begin with the case in which time is discrete and finite, and we then move to the infinite horizon case.
- We also assume that search is **without recall**
- and Search is also **Random/Indirect** which means that you have to search from the entire distribution

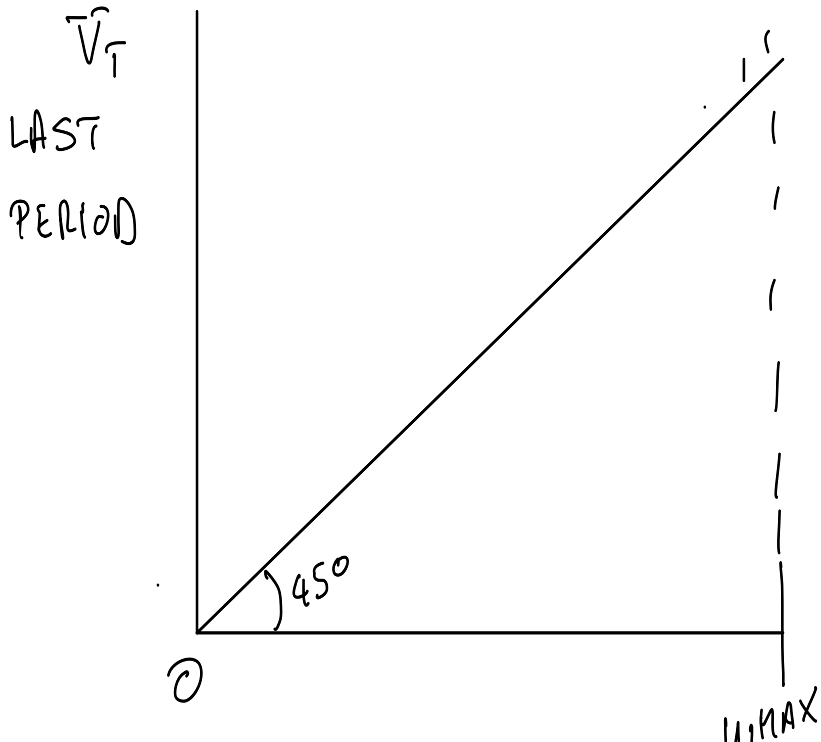
1.1.1 FINITE CASE

- Assume the worker has a discount rate equal to $\beta < 1$
- Assume that worker leaves for T periods.
- Consider an unemployed worker at time t who has just received an offer which has values W.
- The worker faces two choices:
  1. accept the offer W
  2. continue to search and getting an offer next period.
- Let's focus on the problem of an unemployed that has a job/wage W and has already enjoed $B \geq 0$ for this period
  - The goal is to derive/understand the value of continue to search
  - Note that in the course value functions are capital letters
  - If the value of continue to search at time t is U_t (note that U_t is to be determined!)
  - $(V_t|W)$ is the optimal value of having a job at time $t = 1....T$ inluding the option to keey searching.
  - $(V_t|W)$ is thus our value function.
- The problem of the worker is simply solved by

$$(V_t|W) = \begin{cases} Max[U_t, W] & t = 1, 2, 3...t-1 \\ W & \text{if} \quad t = T, \end{cases} \tag{1}$$

- Remember that in the previous equation W is given.
- To solve for the value function we need to determin U_t in each period. Once we have U_t the time t problem is trivially solved.

- In the last period



In last period the value of having the job is $\bar{V}_T = \bar{w}$ since we accept any job!

Figure 1: Last Period: Accept any job

- In the previous periods we claim that the value function is piecewise linear functino.

For $t = 1 \ldots\ldots T-1$

$\exists\ \bar{w}$ such that $\bar{U}_t = \bar{w}$

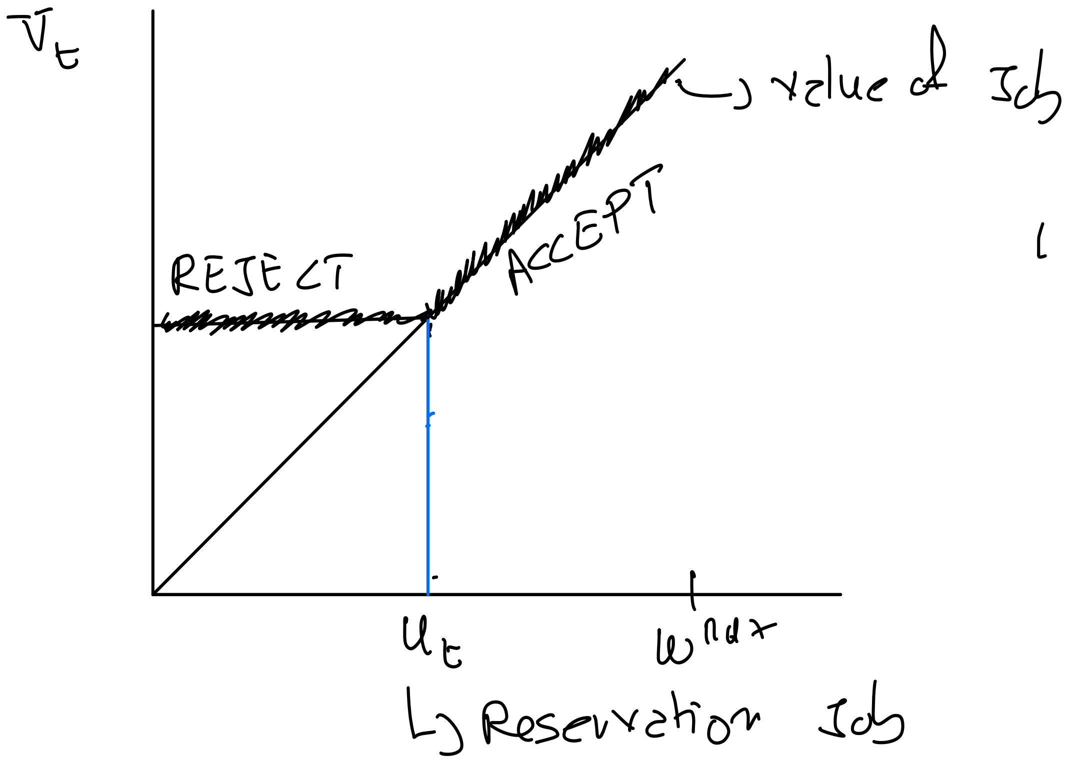


Figure 2: The value function and the reservation wage/job

- The maximization of the agent is whether to accept a W job or continuing to search.
- Clearly, the worker should accept any job that pays at least U_t, so that it is optimal to stop searching if and only if

$$W \geq U_t,$$

  where U_t i*s the reservation offer*, or the value of the job at which the worker is indifferent between working or searching.
- We have learnt two important insights.

  1. First, U_t is the value of a job that makes the worker **indifferent** between working or continuing to search. This maximization problem is described in Figure 61. Given U_t, the value function has a kinked exactly at U_t, which has indeed the interpretation of the reservation offer.
  2. Second, when time is finite the problem is likely to be solved easily, since we know that in the last period of the game the worker will accept each and every job that has received. When time is over, the value of continue to search must be zero.

- The worker has *the option to reject the offer and continue to search*, in which he or she receives b next period in unemployment compensation, and may get a new offer $\tilde{W}$. Let $V_t = W$ the worker's lifetime utility when he has an offer that has value W.
- The problem is to find U_t.
- To find U_t let us imagine of considering a worker who has *just* rejected an offer and is continuing to search. At this point, it should be clear that the value of rejecting the offer and continue to search at time t is

$$U_t = \begin{cases} \beta \left[[b + (1-\pi)U_{t+1} + \pi \left(\int \max[\tilde{W}, U_{t+1}] dF(\tilde{W}) \right) \right] & t = 1, 2, 3...t-1 \\ \\ 0 \qquad \text{if} \qquad t = T, \end{cases} \tag{2}$$

  - If $t = T$ the value of continuing is zero.
  - If $t < T$ than the expression above becomes a proper value function.
  - The right hand side is discounted at rate $\beta < 1$, since everything will happen next period.
  - The next period income is b.
  - Further, a new offer may or may not arrive. The offer arrives at rate π.
    * If the offer does not arrive the worker will continue to search, and will get a value U_{t+1}
    * If an offer arrive how is the problem determined? It is described by the expression in the integral
    * That is simply the problem described by equation 1 above in expected term, where we can get a value $\tilde{W}$ with probability $dF(\tilde{W})$. When will have $\tilde{W}$ on hand, we will do exactly what we did with equation 1 and Figure 1.

- The problem is now clear.
- The worker should in general reject any offer that does not pay at least U_t, which is indeed the value of the reservation offer.
- U_t then solves the discrete time Bellman equation.
- In the finite case, one can solves for the sequence $\{U_1, ..., U_T\}$ by backward induction.
- By working backward, the problem is fully determined. This is very easy to do it in a computer. A technical note: for $\beta < 1$ the right hand side is a contraction mapping by the Blackwell Sufficient conditions for a contraction (See Stokey Lucas).
- Use backward induction
- $U_T = 0$
- Let's now go to $T - 1$

$$U_{T-1} = \beta \left[b + (1-\pi) \overbrace{U_t}^{0}] + \pi \left(\int_0^{W^{Max}} Max[\hat{W}; \overbrace{U_T}^{0} dF(\hat{W}) \right) \right] \tag{3}$$

- This becomes simply once you note that

$$\int_0^{W^{Max}} \hat{W} dF = \int_0^{W^{Max}} \hat{W} f(\hat{W}) d\hat{W} = E(W)$$

$$U_{T-1} = \beta \left[b + \pi E(W) \right]$$

so that U_{T-1} is determined alongside $U_T = 0$

- Once U_{t-1} is determined the optimal value function

$$V_{T-1} = Max(U_{T-1}, W$$

  is immediately determined

- The Optimal Search value function is a **piecewise linear function**

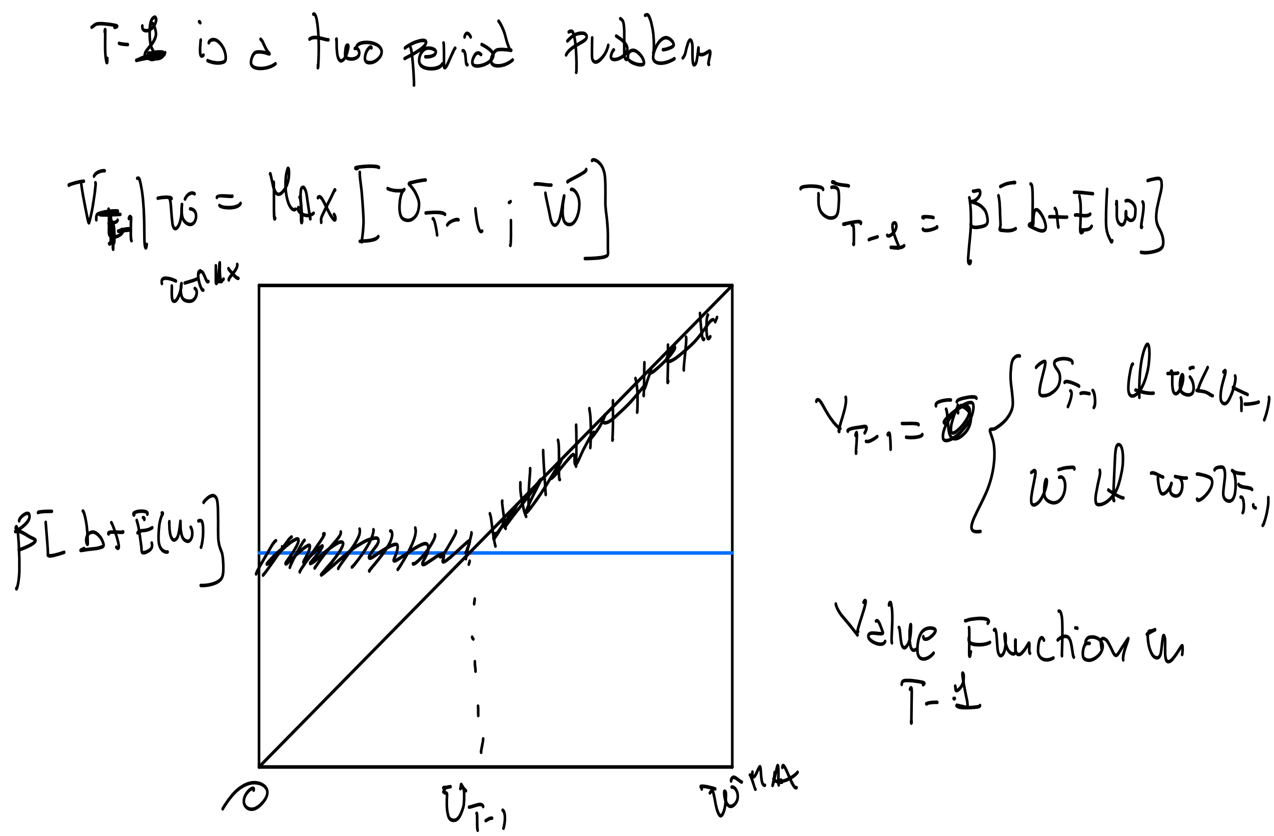


Figure 3: The optimal value function in $T - 1$. a two periods problem

- We can then go to $T-2$

$$U_{T-2} = \beta \left[b + (1-\pi) U_{T-1} + \pi \left(\int_0^{W^{Max}} Max \left[\hat{W}; U_{T-1} \right] dF(\hat{W}) \right) \right] \tag{4}$$

where again given U_{T-1} we can determine U_{T-2}

- The problem is thus solved

$$\{U_1, U_2, ..., \overbrace{U_T}^{0}\}$$

1.1.2 INFINITE HORIZON WITH STATIONARY ENVIRONMENT

- What happen if?

$$T \to \infty$$

- We try to solve for the discrete case and stationary (if it exists) su that In the infinite horizon *stationary* case,

$$\lim_{t \to \infty} U_t = U$$

- In the limit case the reservation value U solves

$$U = \beta[b + (1 - \pi)U + \pi \int_{\tilde{W} \in \Omega} \max[\tilde{W}, U] dF(\tilde{W})] \tag{5}$$

- Note that equation 5 satisfies the reservation property

$$\exists \tilde{W} : U = \tilde{W}$$

so that

$$\forall W \geq U \text{the job is accepted}$$

$$\forall W > U \text{the job is rejected and search goes on}$$

- Note that by using the reservation rule/prperty we can get rid of the Max operatore in equation 5

- The value function U can be further simplified by removing the max operator.
- This is a general property in reservation rule maximization. By virtue of the reservation property the integral in the previous expression can be written as

$$\int_0^{W^{Max}} \max[\tilde{W}, U] dF(\tilde{W})] = \int_0^U U dF(\tilde{W}) + \int_U^{\tilde{W}^{Max}} \tilde{W} dF(\tilde{W})$$

  - in the interval $[0, U]$, U wins the maximization
  - in the interval $[U, W^{Max}]$ all W win the maximization

- Further, since U can be written as

$$U = \int_0^U U dF(\tilde{W}) + \int_U^{\tilde{W}^{Max}} U dF(\tilde{W}) \tag{6}$$

  which simplifies to

$$U = UF(U) + U(1 - F(U))$$

- The reservation rule of equation (5) becomes

$$U = \beta \left[b + (1-\pi)U + \underbrace{\pi \int_0^U U dF(\hat{W}) + \pi \int_U^{W^{max}} \hat{W} dF(\hat{W})}_{\int \max[\tilde{W}, U] dF(\tilde{W})]} \right] \tag{7}$$

  We are left with

$$U = \beta b + \beta U \underbrace{- \beta\pi \int_0^U U dF(\tilde{W})}_{a} \underbrace{- \beta\pi \int_U^{W^{max}} U dF(\tilde{W})}_{b} \underbrace{+ \beta\pi \int_0^U U dF(\tilde{W})}_{c} + \beta\pi \int_U^{\tilde{W}^{Max}} \tilde{W} dF(\tilde{W})$$

  to get (a) and (b) we used 6 and further (a) and (c) cancel each other
  or

$$U = \frac{\beta}{1-\beta} [b + \pi \int_U^{\tilde{W}^{Max}} (\tilde{W} - U) dF(\tilde{W})] \tag{8}$$

  which is an equation that implicitly defines U.

- One has to think of the solution as an implicit a function

$$U = U(b, \beta, \pi, F(.))$$

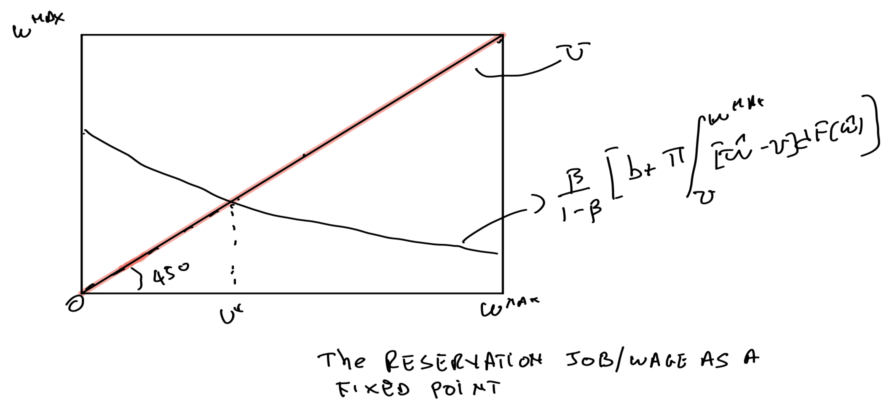


Figure 4: The reservation wage/job

- How can we prove that the RHS of the fundamental equation is downward sloping in U?
  - We need to use implicit differentiation as well as Liebntiz rule in calculus
- Let's recall Liebnitz rule as differentiation of integrals
-

$$g(x) = \int_{a(x)}^{b(x)} f(x;t)dt$$

- We then have

$$\frac{\partial}{\partial x} g(x) = \left[\int_{a(x)}^{b(x)} f(x, b(x)) \frac{\partial b(x)}{\partial x} - f(x, a(x) \frac{\partial a(x)}{\partial x} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x;t)dt \right]$$

- Let's apply Liebnitz rule to the RHS of the reservation productivity

$$U = \frac{\beta}{1-\beta}[b + \pi \int_{U}^{\tilde{W}^{Max}} (\tilde{W} - U)dF]$$

$$\frac{\partial}{\partial U} = \frac{\beta}{1-\beta} \left[-(U - U) + \int_{U}^{W^{Max}} -dF(\hat{W}) \right] =$$

$$\frac{\beta}{1-\beta} - (1 - F(U)] < 0$$

- Two comparative static follows.
-

Remark 1. *An increase in unemployment compensation b increases the reservation offer, so that workers become more choosy*

$$U = \frac{\beta}{1-\beta}[b + \pi \int_{U}^{\tilde{W}^{Max}} (\tilde{W} - U) dF(\tilde{W})] \tag{9}$$

- To see this simply implicitly differentiate equation (9) with respect to b.

$$\frac{\partial U}{\partial b} = \frac{\beta}{1-\beta} - \frac{\beta\pi}{1-\beta}(U-U) - \frac{\beta\pi}{1-\beta}\frac{\partial U}{\partial b}\int_{U}^{\tilde{W}^{Max}} dF(\tilde{W}) \tag{10}$$

Collecting terms in $\frac{\partial U}{\partial b}$

$$\frac{\partial U}{\partial b}[\frac{1-\beta+\beta\pi-\beta\pi F(U)]}{1-\beta} = \frac{\beta}{1-\beta} \tag{11}$$

$$\frac{\partial U}{\partial b} = \frac{\beta}{1-\beta+\pi\beta(1-F(U))} \geq 0 \tag{12}$$

-

Remark 2. *An increase in the arrival rate of offer π increase the reservation offer, so that workers become more choosy*

Differentiate with respect to π yields

$$\frac{\partial U}{\partial \pi}[1 + \beta F(U)] = \int_{U}^{\tilde{W}^{Max}} (\tilde{W} - U) dF(\tilde{W})$$

1.1.3 Continuous Time

- We want to go from discrete to continuous time

- A Stochastic process $\{N(t), t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of "events" that have occurred up to time t.
- A counting process is said to possess *independent increments* if the number of events which occur in disjoint time intervals are independent ($N(10)$ is independent of the event occurring between $N(15)$ and $N(10)$
- A counting process is said to possess *stationary increments* if the distribution of the number of events which occur in any interval of time depends only on the length of the time interval
- The counting process $\{N(t), t \geq 0\}$ is said to be a **Poisson process** having rate $\lambda, \lambda > 0$ if

  1. *i)* $N(0) = 0$;
  2. *ii)* The process has stationary and independent increments;
  3. *iii)* $P\{N(h) = 1\} = \lambda h + o(h)$;
  4. *iv)* $P\{N(h) \geq 2\} = o(h)$.

  Further, the number of events in any interval of length t is Poisson distributed with mean λt. That is for all $s, t \geq 0$

  $$P\{N(t+s) - N(s) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \qquad n = 0, 1, ..$$

- This implies that $P\{N(t) = 0\} = e^{-\lambda t}$ (no events up to time t) has an exponential distribution with mean $1/\lambda$.
- This implies that the assumption of stationary independent increments is equivalent to asserting that, at any point in time, the process probabilistically restarts itself. In other words, the *process has no memory.*

The Asset Value Function

- Let's assume that the interval between time t and time $t+1$ is of length Δ
- Let assume that the arrival rate of job offer follows a Poisson process with arrival rate λ. Equation (1) can be rewritten as

$$U(t) = \frac{1}{1+r\Delta}[b\Delta + (1-\lambda\Delta - o(\Delta))U(t+\Delta) + (\lambda\Delta + o(\Delta)) \int \max[\tilde{W}, U(t+\Delta)]dF(\tilde{W})] \tag{13}$$

where $\beta = \frac{1}{1+r\Delta}$.

- Equation 13 can be written as

$$U(t)r\Delta - [U(t+\Delta) - U(t)] = b\Delta - (\lambda\Delta + o(\Delta))U(t+\Delta) + (\lambda\Delta + o(\Delta)) \int \max[\tilde{W}, U(t+\Delta)]dF(\tilde{W})]$$

  - and diving by Δ yields

$$rU(t) - \frac{[U(t+\Delta) - U(t)]}{\Delta} = b - (\lambda + \frac{o(\Delta)}{\Delta})U(t+\Delta) + (\lambda + \frac{o(\Delta)}{\Delta}) \int \max[\tilde{W}, U(t+\Delta)]dF(\tilde{W})]$$

  - and taking the limit as $\Delta \to 0$

$$rU(t) = b + \lambda[\int \max[\tilde{W}, U(t)]dF(\tilde{W})] - U(t)] + \dot{U}(t) \tag{14}$$

since $\frac{o(\Delta)}{\Delta} \to 0$ as $\Delta \to 0$ and

$$\dot{U}(t) = \lim_{\Delta \to 0} \frac{[U(t+\Delta) - U(t)]}{\Delta}.$$

- $$rU(t) = b + \lambda[\int \max[\tilde{W}, U(t)]dF(\tilde{W})] - U(t)] + \dot{U}(t) \tag{15}$$

- In equation above, $U(t)$ is the "asset" or "option" value of search activity.

- In this interpretation, Equation (2) prices the option by requiring that the opportunity cost of holding it (the left hand side) be equal to the current income flow, plus the expected capital gain flows (the product of the arrival frequency λ and the expected capital gain given an offer arrival), and a pure rate of capital gain or loss attributable to waiting another instant for an offer arrival.

- The value of an optimal search strategy must also satisfy the transversality condition

  $$\lim_{t\to\infty} U(t)e^{-rt} = 0$$

  In the stationary case $U(t) = U$ and the asset value function reads

  $$rU = b + \lambda[\int \max[\tilde{W}, U)]dF(\tilde{W})] - U].$$

- Again, the reservation value function can be written without max operator as

  $$rU = b + \lambda[\int_U^{W^{MAX}} [\tilde{W} - U]dF(\tilde{W})$$

- If $\tilde{W}$ is degenerate and can take only 1 possible value W and if the labor market is *viable* ($W > U$) the asset function reads

  $$rU = b + \lambda[W - U]$$

  which is the simplest value function we will be playing with, and the one that is used in the baseline matching model (see Handout #2).

2 From a Micro Problem to the First Model of Unemployment

- Let's turn this pure micro problem in a first model of unemployment
- Let's assume many identical individuals behave according to the **Discrete** version of the Mc Call 1 sided search.
- The reservation job/value in a stationary environment is

$$U_t = R_t = R$$

so that

$$R = \frac{\beta}{1-\beta}\left[b + \pi \int_{R}^{W^{max}} \left[\hat{W} - R\right]\right] dF(\hat{W}) \tag{16}$$

assume $\pi = 1$ so that you get a job offer every single period.

- Each period
  - A Mass $s > 0$ of workers enter the labour market as unemployed
  - jobs last forever but people/workers die at rate s
- u_t is the stock of unemployed at time t
- Accounting dynamics of u_t in this economy.

$$u_{t+1} = \underbrace{s}_{New\ \ entrants} + \underbrace{F(R)}_{\text{Probability Rejection}} \underbrace{(1-s)}_{Survivors} u_t \tag{17}$$

- Since

$$F(R)(1-s) < 1$$

$$\exists u^* : u_{t+1} = u_t$$

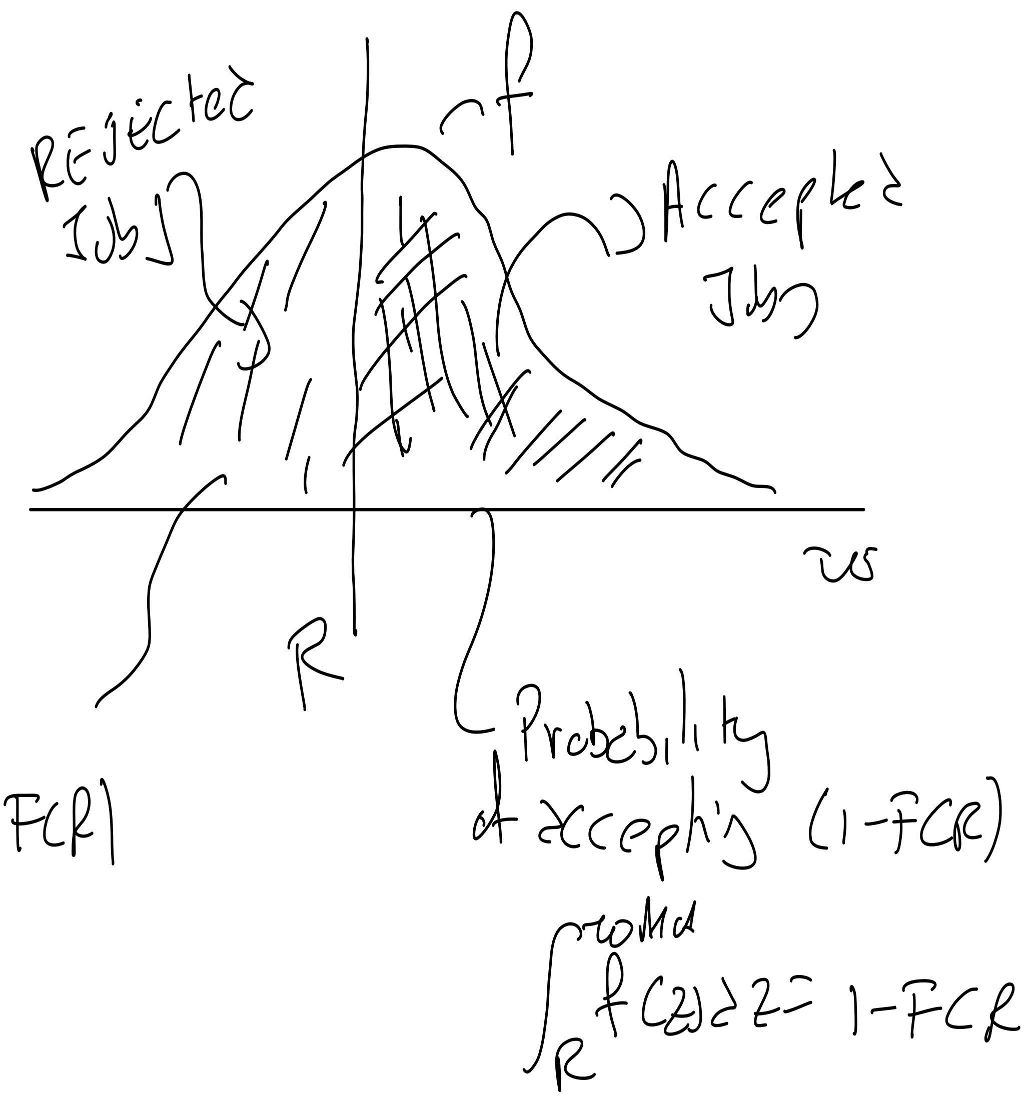


Figure 5: Accepted Jobs each period

- Let's work with the dynamics of unemployment
  -
$$u_{t+1} = \underbrace{s}_{New \quad entrants} + \underbrace{F(R)}_{\text{Probability Rejection}} \underbrace{(1-s)}_{Survivors} u_t \tag{18}$$
  - Subtracting u_t on both sides and adding and subtracting su_t
$$\underbrace{u_{t+1} - u_t}_{0 \quad in \quad steady \quad state} = s + F(R)(1-s)u_t - u_t \pm su_t \tag{19}$$
  - Calcolating in s.s that we know to exist
$$0 = s + F(R)(1-s)u^* - su^* - (1-s)u^* \tag{20}$$
- so that the steady state level of u_t in 1 sided search is
$$u^* = \frac{s}{s + (1-s)(1-F(R))} \tag{21}$$
- Let's think about the implications.
  - Assume $F(R) = 1$ so that all jobs are rejected. Equilibrium unemployment becomes
$$u^*(F(R) = 1) = \frac{s}{s} = 1$$
  - Assume now that $F(R) = 0$ so taht all jobs are accepted
$$u^*(F(R) = 0) = \frac{s}{s + (1-s)} = s \tag{22}$$
    so that the only unemployed are the new entrants. They spend one period unemployed and then transit into a permanent job

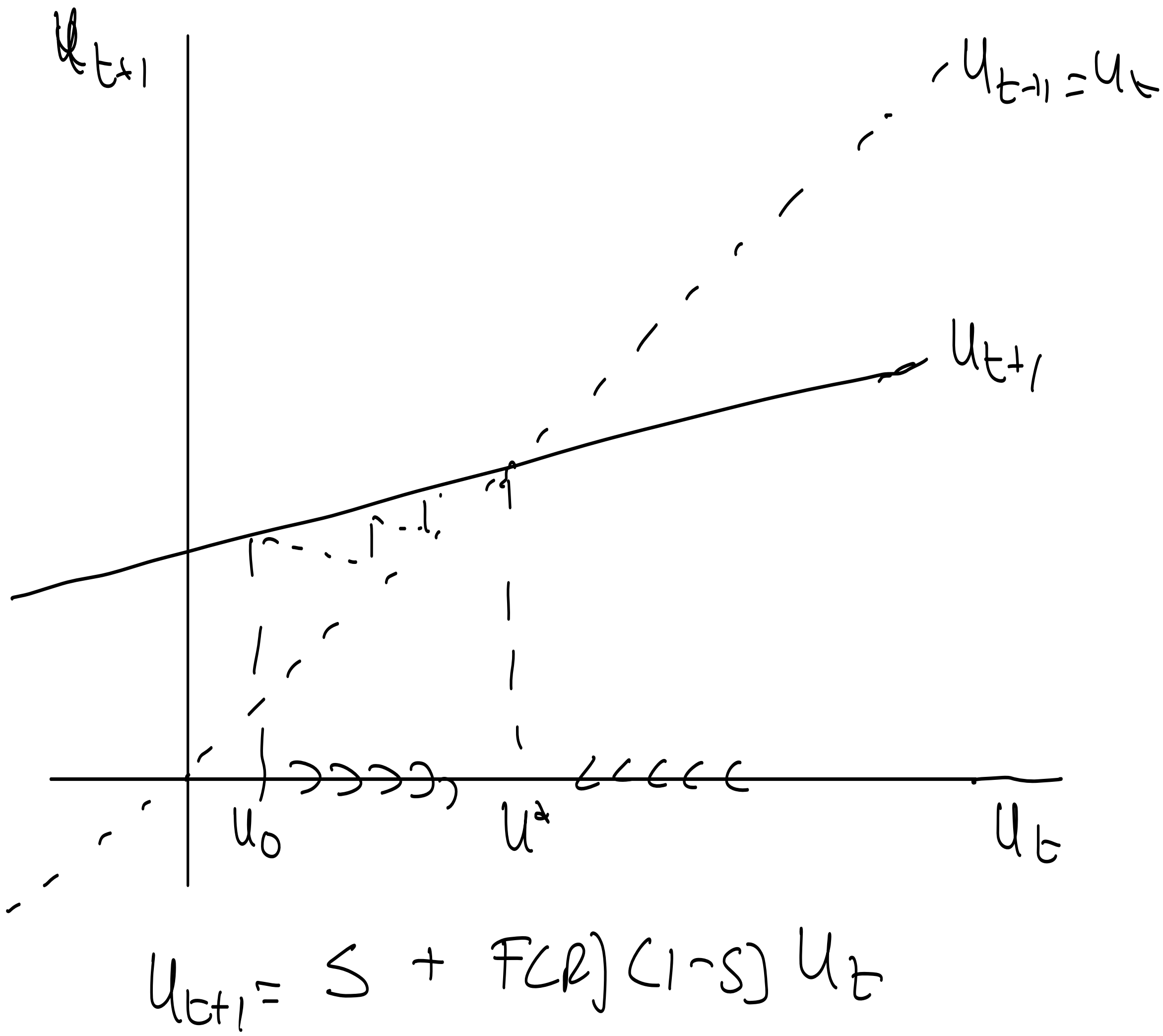


Figure 6: Unemployment Dynamics as a Difference Equation

2.1 Discussion

- This is a real microfounded model of unemployment.
- It is indeed a search based model of the NRU.
- Yet, we do not like it.
- What are the problems of this model ?
  - Where the distribution $F(\hat{W}))$ come from ?
  - F is completely exogenous in the model.
  - Firms are completely absent.
  - By introducing firms into this environment scholars end up in the so called **DIAMOND PARADOX**
- What is the Diamond Paradox?
  - One can show that the distribution of wage offers must have a mass point
  - The only Nash Equilibrium in one in which $R = 0$ (Full monopsony power)
  - Look at Acemoglu notes for a formal proof of the paradox.
- We thus move to two sided search and the matching model of unemployment

2.2 Appendix: Solving the time-varying value function

- This is useful for solving and integrating forward time varying value functions.
- It is basic like solving a differential equations

$$(\delta + r)J(t) = p(t) - w(t) + \frac{d}{dt}J(t) \tag{23}$$

Rewrite it as

$$\dot{J} - (\delta + r)J(t) = -p(t) + w(t)$$

where $\dot{J} = \frac{d}{dt}J(t)$. Multiplying both sides by $e^{-(\delta+r)t}$ yields

$$e^{-(\delta+r)t}[\dot{J} - (\delta + r)J(t)] = e^{-(\delta+r)t}[-p(t) + w(t)] \tag{24}$$

Noting that in the LHS

$$e^{-(\delta+r)t}[\dot{J} - (\delta + r)J(t)] = \frac{\partial}{\partial t}[e^{-(\delta+r)t}J(t)]$$

we can integrate from to t up to infinite both sides of equation 24to yield

$$\int_t^\infty \frac{\partial}{\partial s}e^{-(\delta+r)s}J(s)ds = -\int_t^\infty e^{-(\delta+r)s}[p(s) - w(s)]ds$$

$$\left| e^{-(\delta+r)s}J(s) \right|_t^{\lim s\to\infty} = -\int_t^\infty e^{-(\delta+r)s}[p(s) - w(s)]ds$$

or simply

$$\lim_{s\to\infty} e^{-(\delta+r)s}J(s) - J(t)e^{-(\delta+r)t} = -\int_t^\infty e^{-(\delta+r)s}[p(s) - w(s)]ds$$

Using the transversality condition

$$\lim_{s\to\infty} e^{-rs}J(s) = 0$$

the value function reads

$$J(t) = e^{-(\delta+r)t}\int_t^\infty e^{-(\delta+r)s}[p(s) - w(s)]ds$$

or

$$J(t) = \int_t^\infty e^{-(\delta+r)(s-t)}[p(s) - w(s)]ds$$