The Baseline Matching Model: Job Creation

[Sem0057]

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Academic Year 2024-2025

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1 Introduction

- What is a Matching Function/Technology
  - It is a technology that describes how meeting take place in the market
  - It is somewhat similar in spirit to the production technology that describes how combined input generate output.
  - The key idea in the SAM literature is that matching technology can be described by a simple aggregate functions of few variables.

2 The Matching Function

- Production Function describes how input are combined to obtained a given output. It is a tool box of any introductory econ student
- Matching Function. it describes how **meetings** take place among market participants. It is a key tool box for any SAM scholar and student.
  - It is a bit of black box as much as the production function is in production theory.
  - The key idea is that meetings among market particiapnts are a time consuming process.
  - The number of matches per unit of time is mL, (most of the time $L = 1$ is the size of the labor force)
  - m is thus the number of meeting per unit of time
  - time is continuous and we are in stationary environment
- Key assumption: mL of m depends on very few variables
- There are two types of searching agents. We focus on the labor market
  - **Workers**. The unemployed is a worker searching for a job
  - **Firms**. A Vacant firms is a firm searching for a worker.

- The Basic Matching Function

$$mL = m(uL, vL)$$

with

  - u is the stock of unemployed
  - v is the stock of vacancies.
  - So that meetings depends on the absolute number of searchers.
  - u is a measure of job seekers' effort (read unemployment rate) while v is a measure of firm's recruiting effort (read vacancy rate, the number of vacancies normalized by the labor force).

- Basic Assumptions

$$m < min(u, v)$$

which implies that somebody goes "unmatched" in a unit of time.

- Typically there are more assumptions on the matching function

  - Positive first derivatives:
  $$m_u > 0; \quad \text{and} \quad m_v > 0$$
  it makes a lot of sense. The more searcher of one ype, the more meeting in the markets
  - Negative Second derivatives
  $$m_{uu} < 0; \quad \text{and} \quad m_{vv} < 0$$
  it is in line with diminishing returns in economics. At the margin the contribution of additional searchers falls.
  - There are CRS to matching
  $$\Delta m = m(\Delta u, \Delta v)$$
  so that the size of the market is irrelevant (a nice feature for cross country comparison). CRS is also "empirically plausible".
  - Sometime we also assume Inada conditions.
  $$lim_{u \to 0} m_u = lim_{v \to 0} m_v = \infty$$
  and
  $$lim_{u \to \infty} m_u = lim_{v \to \infty} m_v = 0$$

Transition Rates

- Given the function m and its asumption we can derive transition rates
- Let0s take the firm standpoint. What is the probability that a vacant firm meets an unemployed.
- Let $q()$ be the rates at which vacancies are filled

$$q() = \frac{m(u,v)}{v} = m(\frac{u}{v}, 1)$$

- The key variable in the matching enviroment is **Market Tightness**

  Let

$$\theta = \frac{v}{u}$$

  so that Think of market tightness as to how tight is the market form the firm stand point. If θ is high, the search environment is such that there are many searcher of the firm type

- The probability that a firm meets a worker - GIVEN CRS- depends only on θ

$$q(\theta) = m(\frac{u}{v}; 1) \qquad q'(\theta) < 0$$

- We can make the Cobb douglas case right away. If Cobb-Douglas

$$m(u,v) = v^{1-\alpha}u^{\alpha}$$

  and

$$q(\theta) = \frac{v^{1-\alpha}u^{\alpha}}{v} = (\frac{v}{u})^{-\alpha} = \theta^{-\alpha}$$

- The rates at which workers find jobs is

$$\alpha(\theta) = \frac{m(u,v)}{u} = \frac{v}{u}\frac{m(u,v)}{v} = \theta q(\theta) \qquad\qquad \alpha'(\theta) > 0$$

  - Since

$$\alpha'(\theta) = q(\theta) + q'(\theta)\theta =$$

  - and collecting $q(\theta)$

$$= q(\theta)[1 + \frac{q'(\theta)\theta}{q(\theta)}] = q(\theta)[1 - \eta(\theta)] > 0$$

    where

$$\eta(\theta) = \epsilon_{q,\theta} \left| \frac{q'(\theta)\theta}{q(\theta)} \right| < 1$$

  - with CRS in matching the elasticity of the matching function with respect to θ is less than one in absolute value.

- With a Cobb Douglas matching function

$$m(u,v) = u^{\alpha} v^{1-\alpha}$$

- Given market tightness $\theta = \frac{v}{u}$

$$q(\theta) = \left(\frac{u^{\alpha} v^{1-\alpha}}{v} \right) = \left(\frac{u}{v} \right)^{\alpha} = \theta^{-\alpha}$$

- Note that

$$q'(\theta) = -\alpha\theta^{-\alpha-1} < 0$$

- With CObb DOuglas the elasticity $\eta(\theta)$ is also independent of θ and is equal to α.

$$\alpha(\theta) = \theta^{(1-\alpha)}$$

- and

$$\eta(\theta) = \left| -\alpha\theta^{-\alpha-1} \frac{\theta}{\theta^{-\alpha}} \right| = \alpha$$

2.1 Discussion and Implications

- How should $q(\theta)$ be interpreted?
  - $q(\theta)dt$ is the arrival rate of offer, a Poisson Process with arrival rate $q(\theta)$, so that $q(\theta)dt$ is the probability that a firm meets a workers
  - Note that $0 \leq q\theta) \leq +\infty$ since arrival rate can be infinitly large.
  - What has to bo bounded is the probabilty tha ta firm meets a worker

$$0 \leq q(\theta)dt \leq 1$$

- There is **stochastic rationing** (can not be eliminated by price adjustment):

$$(1 - q(\theta)dt) > 0$$

and (

$$1 - q(\theta)dt) > 0$$

.

- There are **trading externalities** (transition rates depends on the relative numbers of agents
  - If one type of the opposite side of the math enter the market, the probability of being mathced increases (we will call it the thick/thin market externality).
  - If one type of the same side of the math enter the market, the probability of being mathced increases. (we will call it the congestion externality)
  - THis is very important in the mating game in nature. Also relevant in marriage markets
  - This will be key as weill speak about efficeincy.

2.2 Unemployment Flows

- Labor force is constant and normalized to 1

$$u + n = 1$$

where u is the unemployment rate and n is employment.

- The labor force is thus fixed. Remember that the Working Age Population is defined as the population between the age of 16 and 64 that cna be in one of three states

$$WAP = u + n + o \tag{1}$$

where o is the stock of people out of the labor force. We are thus assuming $o = 0$. Garibaldi and Wassmer (2005) conside a mathcing model with also flow in and out of the labor force.

- In the matching model the stock of unemployment is derived by the flows. It is the famous "bath-tab" approach to stock. When is the bath water constant? When inflows are equal to outflows.
  - Steady state unemployment: inflows=outflows.
- We need some assumptions on unemployment inflows.
- In the simple model unemployment inflows are equal to total job destruction.
  - Jobs are destroy at an exogenous rate equal to λ, where λ is the arrival rate (Poisson) of idiosyncratic shocks.
  - sdt is the probability of destruction.
  - Total Job Destruction

$$JD = \lambda(1 - u)$$

- How many meeting take place in a given time interval?
  - How about unemployment outflows?
  - The simple idea is that **any meeting** in the labor market leads to a new jobs.
  -

$$JC = \theta q(\theta) u$$

    which corresponds to the number of meeting formed by the matching function.
  -

$$JC = \frac{m(u, v)}{u} u = \theta q(\theta) u$$

- There is no on the job search

- What is the dynamics of unemployment

$$\dot{u} = \frac{du}{dt} = \text{Unemployment Inflows} - \text{Unemployment outflows}$$

$$\dot{u} = \frac{du}{dt} = \text{Job Destruction } - \text{Job Creation}$$

- Unemployment is constant if

$$uq(\theta)u = \lambda(1-u)$$

which implies

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}, \tag{UV-KEY1}$$

where θ is one of the key endogenous variables and λ is an exogenous parameter.

- This is a negative relationship between u and θ. Alternatively, in the $u-v$ space the unemployment relationship is

$$\lambda(1-u) = x(u,v)$$

.

This is called **Beveridge curve**.

- The **Beverdige Curve** is the combination of $v-u$ for which unemployment is constant.

Totally differentiating yields

$$-\lambda du = x_u du + x_v dv$$

which implies

$$\frac{dv}{du} = -\frac{\lambda + x_u}{x_v} < 0$$

- Is it convex?

$$\frac{d^2v}{du^2} = -\frac{\overbrace{m_{uu}m_v}^{-\times+=-} - \overbrace{m_{vu}m_u}^{+\times+=+}}{m_{vv}^2} = -\frac{-}{+} = + \tag{2}$$

it is convex if $m_{uv} > 0$ so that the cross derivative of the vacancy and unemployment rate is positive.

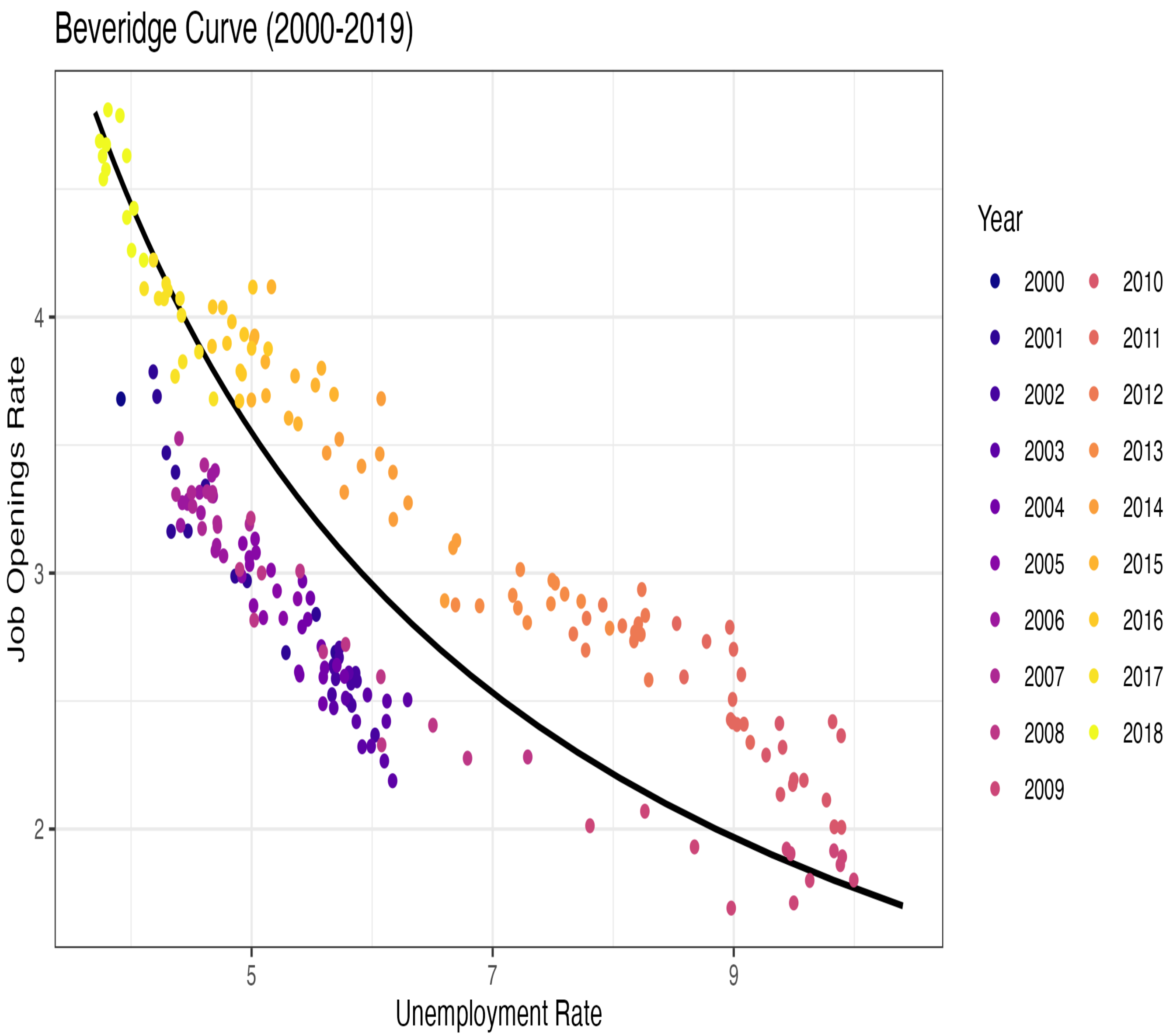

Figure 1: The Actual Beveridge CUrve from the US

- We now have the first key equation of equilibrium unemployment theory.

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}, \tag{UV-KEY1}$$

  where θ is one of the key endogenous variables and λ is an exogenous parameter.

- From the previous equation we need three elements

  1. λ is an exogenous parameter the govern job destruction
  2. The function $q()$ is the matching function
  3. Market tightness $\theta = \frac{v}{u}$ is the key endogenous variable

- Once we have θ we can get unemployment from the beveridge curve as $u = u(\theta)$

- The basic model derives θ from value functions for workers and firms

3 The Basic Model

3.1 Firm Behaviour

- Firms and workers are risk neutral, and discount the future by $r > 0$.
- The environment is stationary and time is continuous
- At the firm level, the arrival rate of offer follow a Poisson process with rate $q(\theta)$.
- The per worker productivity is indicated with p. Think of a CRS technology, so that p is the output of a 1 firm/worker job match
- p is a flow value.
- A job can be in two states

  1. Filled job, and we indicate the PDV (Present discounted value) as J
  2. Vacant job, and we indicated the PDV as V

  Firms are one firm-one job

- w is a flow instantenous payment from the firm to the worker. It is basically the wage

- The value function of a single job is

$$rJ = p - w + \lambda[V - J] + \overbrace{\dot{J}}^{0 \quad \text{in stationary}}$$

which implies

  - Think of λ shock hitting the firm as huge negative shock for which the productivity p falls from p to a larget negative number.

- V is the value of a vacancy
- Search is costly to the firm and it costs $pc > 0$
- The value of a vacancy reads

$$rV = -pc + q(\theta)[J - V]$$

where pc are search costs

- We need a notion of free entry. $V > 0$ gives firm incentive to enter the market
  - Free entry in the job market implies

$$V = 0$$

which corresponds to a key long-run zero profit condition.

  - Which implies

$$0 = -pc + q(\theta)J$$

- Key implication of free entry

$$V = 0; \quad \text{Free Entry} \tag{3}$$

$$J = \frac{pc}{q(\theta)}; \quad \text{Filled Job must have a positive value} \tag{4}$$

  - pc is a flow cost
  - $q(\theta)$ is the arrival rate of workers to firms
  - $\frac{1}{q(\theta)}$ is the average waiting time of vacancy, or the average duration of a vacancy

  so that

$$\underbrace{J}_{\text{Value of Filled Job}} = \underbrace{\frac{pc}{q(\theta)}}_{\text{Expected Search Costs}} \tag{5}$$

  which implies that the value of a job is equal to the search costs.

- The value of a filled job is

$$rJ = p - w + \lambda[\overbrace{V}^{=0} - J]$$

$$J = \frac{p - w}{r + \lambda} \geq 0$$

where $p - w$ are operational profits and $r + \lambda$ is the firm's effective discount rate.

- Think about it

$$\underbrace{\frac{p - w}{r + \lambda}}_{\text{value of a filled job}} = \underbrace{\frac{pc}{q(\theta)}}_{\text{Total Search Costs}} \tag{6}$$

- THis has very important implications in terms of sunk costs and the dynamic life of a job. It is indeed true that a filled job has positive value, but simply becauase the search costs are sunk.

- Substituting this into equation 5 yields

$$w = p - \frac{(\lambda + r)pc}{q(\theta)} \qquad \text{(LABOR DEMAND)}$$

which is a sort of labor demand and downward sloping in a $w - \theta$ space

- It can be described as a generalized labor demand. It it downward sloping?

$$\frac{\partial w}{\partial \theta} = + \frac{\overbrace{q'(\theta)}^{-} pc(r + \lambda)}{q(\theta)^2} < 0$$

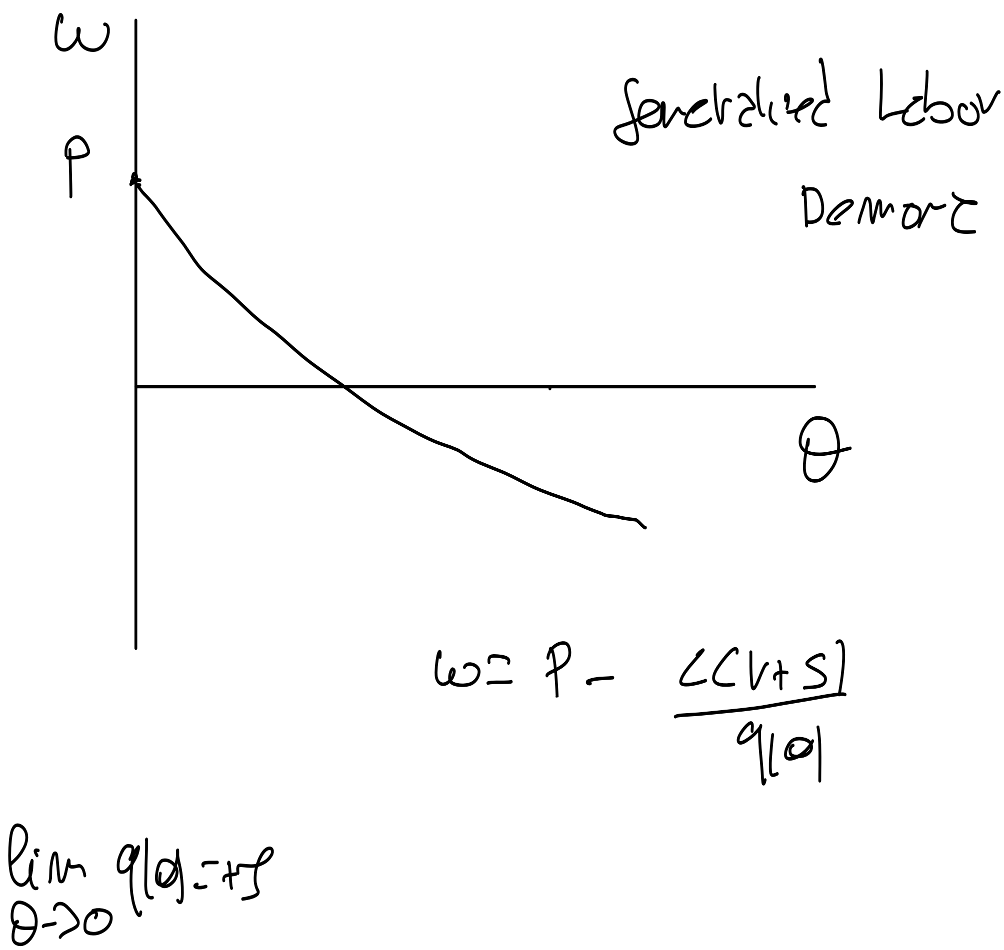

Figure 2: The Generlized labor demand

3.2 Worker Behavior

- U value of unemployment (the PDV)
- W is the value of employment (the PDV)
- Let's start from 1-sided search in hte Mac Call Model

$$rU = b + \pi \int_{U}^{W^{Max}} \left[\hat{W} - U\right] dF(\hat{W})$$

  - Assume that the wage distribution is degenerate, so that there is only one type of job.
  - The model becomes

$$rU = b + \pi \left(W - U\right); \quad \text{if } W \geq U$$

- At the firm level, the arrival rate of offer follow a Poisson process with rate $q(\theta)$.
- At the worker level, the arrival rate of offer follow a Poisson process with rate $\theta q(\theta)$, so that $\pi \to \theta q(\theta)$
- An unemployed worker enjoys an income z and its value of search reads

$$rU = b + \theta q(\theta)[W - U] \tag{7}$$

  where W is the P.D.V. value of having a job while U is the P.D.V. value of unemployment.

- The value of a job is

$$rW = w + \lambda[U - W] \tag{8}$$

- We say that a labor market is viable if $W \geq U$
  - Which jobs should we accept? Amy job such that

$$W \geq U$$

3.3 Participation Constraints

- We said that worker should accept jobs such that

$$W \geq U$$

  - This suggests -intuitively- that there is a minimum acceptance wage such that

$$w_{min} \Longrightarrow (W - U) = 0$$

  -

$$rU(w_{min}) = b + \theta q\theta \overbrace{(W(w_{min}) - U(w_{min})}^{0} \tag{9}$$

  -

$$rU(w_{min}) = b$$

  and evaluated at such value we have that

$$rW(w_{min}) = w_{min} + 0$$

  so that

  -

$$w_{min} = b$$

  and

$$\forall w \geq b \qquad \text{worker accept the job}$$

  - As long as the market delivers b, the worker participates.

- Let's turn to the firm maximum acceptance wage.
-
$$J \quad \text{and} \quad \overbrace{V}^{0} \geq 0$$

  - Since free entry implies that $V = 0$, the firm participation constraint is
$$J \geq 0$$
  - We define w^{max} such that
$$w^{max} \Longrightarrow J(w^{max}) = V = 0$$
  - Using the firm asst equation
$$rJ(w^{max}) = p - w^{max} + \lambda \overbrace{(V - J(w^{max}))}^{0} \tag{10}$$
    which implies
$$w^{max} = p$$
  - From the firm standpoint the market is viable iff
$$w: \quad w \leq w^{max} = p$$

- Putting together firm and worker participation constraint we have
$$b \leq w \leq p$$

- **Labor Market is viable** if
$$p \geq b$$

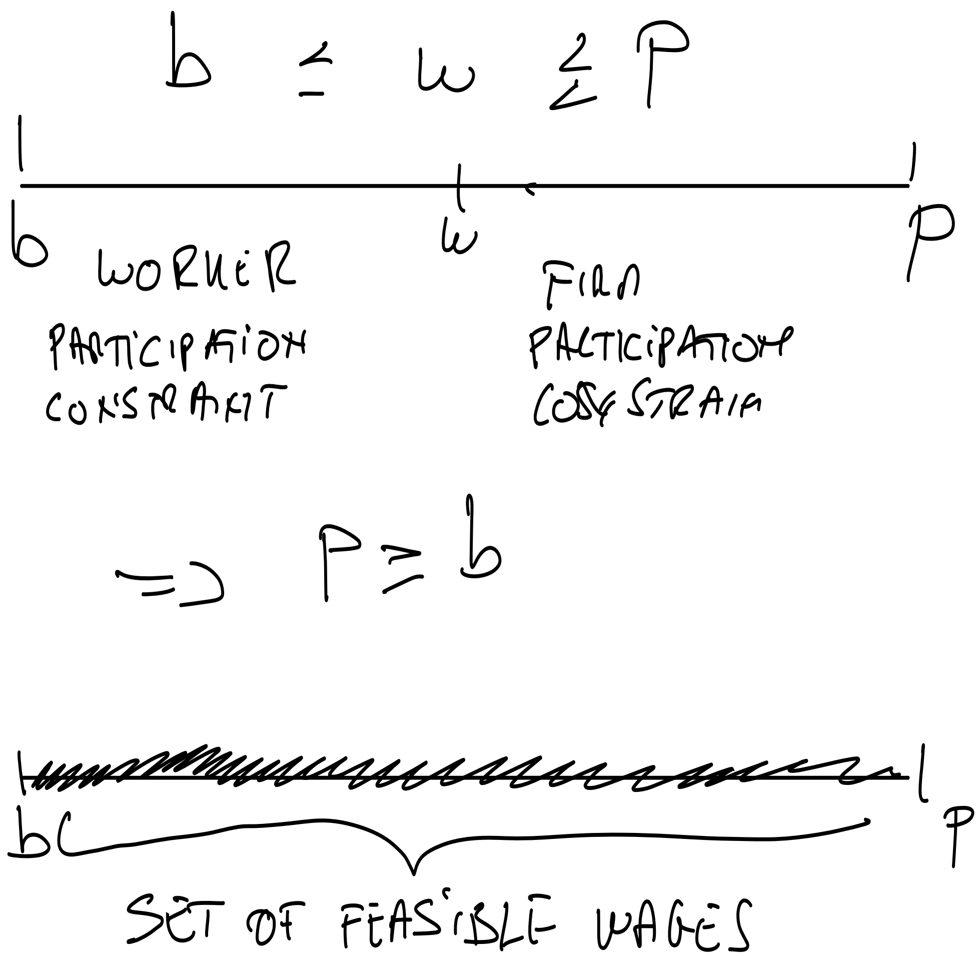

Figure 3: The Set of Feasible Wages

3.4 Wage Determination-NASH BARGAINING

- The discussion on worker participation suggests that there is a continuum of wages that are coherent with the participation constraint.
- We are thus missing 1 equation for uniquely determine w

$$w = p - \frac{(r+\lambda)pc}{q(\theta)} \qquad \text{Job creation}: \quad w \leq p \tag{11}$$

$$rU = b + \theta q(\theta)\,[W-U] \qquad \text{Worker Search Value}: \quad w \geq b \tag{12}$$

$$rW = w + \lambda[U-W] \tag{13}$$

- This is the sense of **wage indeterminacy** that was suggested by Dale Mortensen in the 80s at the time when the idea of equilibrium unemployment theory was developing.
- There is a **pure economic rent**.
  - Where does the rent come from? It comes from the fact that a given paired match (a firm and a worker) have the option to form a job once having paid the search costs and being successfully paired together. If they split, the have to start again. Thus, they are sitting on a rent.
- Matched firms and workers have local monopoly power. Technically, they have a bilateral monopoly problem. In micro we know hot solve, competition, monopoly, monopsony, oligopoly, but we have problems with bilateral monopoly. How do you split a cake?

- In the baseline model wages are obtained through the outcome of a generalized Nash-bargaining between the **firm and the worker after they have met**.
- It was introduced by Mortensen and Diamond in the mid/early 80s
- The NBW **Nash Bargaining Wage**
  - it is very elegant
  - it is partly microfounded and linked to game theory
  - Yet, the NBW has quantiative problems. We will see that it can not match the US Business cycle properties (**Shimer critique**)
- there are indeed many alternatives to the NBW
  - Hall argued that **any** wage within the bragaining set is good enough
  - Hall Milgrom propose Credible Bargaginig

The Surplus

- We need to introduce the concept of the surplus
- The surplus for the firm of having a job is

$$S_\Pi = J - \overbrace{V}^{0}$$

- THe surplus of the worker is

$$S_W = W - U$$

- and the participation constraints can be seen as

  –

  $$S_\Pi(w) \geq 0; \quad J \geq V$$

  –

  $$S_W(w) \geq; \quad W \geq U$$

- What is the Surplus from the job ?

$$S = S_\Pi + S_W = (J - V) + (W - U)$$

or

$$S = J + W - U; \quad \text{Pure Economic Rent}$$

and it is a bilateral monopoly problem

- The axiomatic approach to Nash Bargaining
- We look for the Generalized NBW with parameter $\beta \in [0,1]$
- We follow and can show that that wages are obtained as the solution to

$$w = \arg\max (W-U)^{\beta}(J-V)^{(1-\beta)}$$

where $0 \leq \beta \leq 1$ is the worker's bargaining strength.

- This results is due to John Nash in the 50s. (see Obsborne and Rubinstein)
-

Claim 1. *The NBW solution implies a simple wage rule*

$$W - U = \beta S$$

- To check the claim let's go back to the value functions

$$(r+\lambda)J = p - w \tag{14}$$

so that

$$\frac{\partial J}{\partial w} = -\frac{1}{r+\lambda}$$

- The value of a vacancy is

$$(r+q(\theta))V = -cp + q(\theta)J^e$$

where note that in the rhs J^e is meant to signal that when a vacancy is posted the job in the right hand side is the "average" value of a job and it is not affected by the current wage negotiation. This is subtle but important. With this caveat in mind on has

$$\frac{\partial V}{\partial w} = 0 \tag{15}$$

- Turning to workers one has

$$(r+\lambda)W = w + \lambda U$$

so that

$$\frac{\partial W}{\partial w} = \frac{1}{r+s}$$

and

$$rU = b + \theta q(\theta)\left[W^e - U\right]$$

so that the unemployed worker face the average job in the market that is not affected by the current wage negotiation, so that

$$\frac{\partial U}{\partial w} = 0$$

- To obtain the wage first take the log of the Nash-maximand

$$\Omega = (W-U)^{\beta}(J-V)^{(1-\beta)}$$

so that $ln\Omega = (1-\beta)lnJ + \beta ln(W-U)$

and take the derivative with respect to w.. So that

$$\frac{\partial}{\partial w}\ln\Omega = 0$$

or

$$\frac{\beta}{W-U}\frac{\partial W}{\partial w} + \frac{1-\beta}{J-V}\frac{\partial J}{\partial w} = 0$$

and

$$\frac{1-\beta}{J}\frac{\partial J}{\partial w} + \frac{\beta}{W-U}\left[\frac{\partial W}{\partial w} - \overbrace{\frac{\partial U}{\partial w}}^{0}\right] = 0$$

- Since $(\lambda + r)\frac{\partial W}{\partial w} = 1$ and $(\lambda + r)\frac{\partial J}{\partial w} = -1$, we have

$$(W-U)(1-\beta) = \beta J \quad \text{Sharing Rule} \tag{16}$$

or

$$W - U = \beta\underbrace{(J - V + W - U)}_{S}$$

where $S = J - V + W - U$ is the total surplus.

- From the Sharing Rule to the wage equation

$$W - U = \beta S \tag{17}$$
$$J = (1-\beta)S \tag{18}$$

- How How do you go form the sharing rule to the wage?
- To get the wage proceed as follows. Get the two value "on the job" so that

$$(r+\lambda)J = p - w \tag{19}$$
$$(r+\lambda)W = w + \lambda U \tag{20}$$

and recall

$$rU = b + \theta q(\theta)[W - U]$$

- We introduce a useful concept in search theory that is beyond the surplus, it is ith **joint income of the match**

$$S = J - V + W - U; \quad \text{Joint Surplus} \tag{21}$$
$$M = J + W; \quad \text{Joint Income from the Match} \tag{22}$$

- We can get the value function for M as

$$(r + \lambda)(J + W) = p + \lambda U \tag{23}$$

the joint income is an abstract concept but it is very useful because it has no wag in its expression (since it cancel out)

and of course

$$S = M - U$$

- so that

$$(r + \lambda)(J + W - U) \quad = p + \lambda U - (r + \lambda)U \tag{24}$$
$$(r + \lambda)S \quad = p - rU \tag{25}$$

- this is a deep expression since it says that surplus is

$$S = \frac{p - rU}{r + s} \tag{26}$$
$$S = \int_t^{\infty} (p - rU)e^{-(r+\lambda)dt} \tag{27}$$

the **cake/rent/surplus is thus the PDV of the difference between the value of productiivty and the permanent income of the unemployed!**

- but then

$$J = \frac{p - w}{r + \lambda} \qquad J = (1 - \beta)S$$

so that we can say

$$p - w = (r + \lambda)J \tag{28}$$
$$= (r + \lambda)(1 - \beta)\frac{p - rU}{r + \lambda} \tag{29}$$
$$p - w == (1 - \beta)(p - rU) \tag{30}$$

- We arrive to the first key equation for the wage

$$w = (1 - \beta)rU + \beta p \quad \textbf{NBW version 1} \tag{31}$$

- Is there a way to get rid of U? Yes, the value function is endogenous in the standard representation of the equilibrium
- Remember that unemployed income is

$$rU = b + \theta q(\theta) \underbrace{[W - U]}_{\beta S}$$

- also

$$W - U = \beta S; \qquad J = (1 - \beta)S$$

and since

$$J = \frac{cp}{q(\theta)}; \implies \quad (1 - \beta)S = \frac{cp}{q(\theta)}; \implies \quad S = \frac{cp}{(1 - \beta)q(\theta)}$$

- So that unemploymend income reads

$$rU = b + \theta q(\theta)\beta \frac{cp}{(1 - \beta)q(\theta)}$$

or

$$rU = b + \frac{\beta c\theta}{1 - \beta}; \quad \text{Equilibrium Expression for } rU$$

- This can be substitute into the wage espression

$$w = (1 - \beta)rU + \beta p \quad \textbf{NBW version 1} \tag{32}$$

- So that

$$w = (1 - \beta)\left[b + \frac{\beta c\theta}{1 - \beta}\right] + \beta p$$

$$w = b(1 - \beta) + \beta c\theta + \beta p$$

- yields the final expression for the wage

$$w = b(1 - \beta) + \beta p(1 + c\theta) \qquad (\textbf{ Pure Wage Equation})$$

where $pc\theta$ is the average hiring cost per unemployed

3.4.1 EQUILIBRIUM

- **Definition 1.** ***The Search equilibrium with exogenous job destruction is a triple*** (u, θ, w) *satisfying*

  -
    - *Free entry* ($V = 0$)
    - *Wage Bargaining* $(1-\beta)[W-U] = \beta J$
    - *Balance Flow* ($m(u, v) = \lambda(1-u)$)

  The reduced form is obtained by the three equations

$$\begin{aligned} w &= p - \frac{(\lambda + r)pc}{q(\theta)} \\ w &= b(1-\beta) + \beta p(1 + c\theta) \\ u &= \frac{\lambda}{\lambda + \theta q(\theta)} \end{aligned}$$

- The system is recursive. The wage equation and the free entry gives w and θ. The Beveridge curve gives u given θ

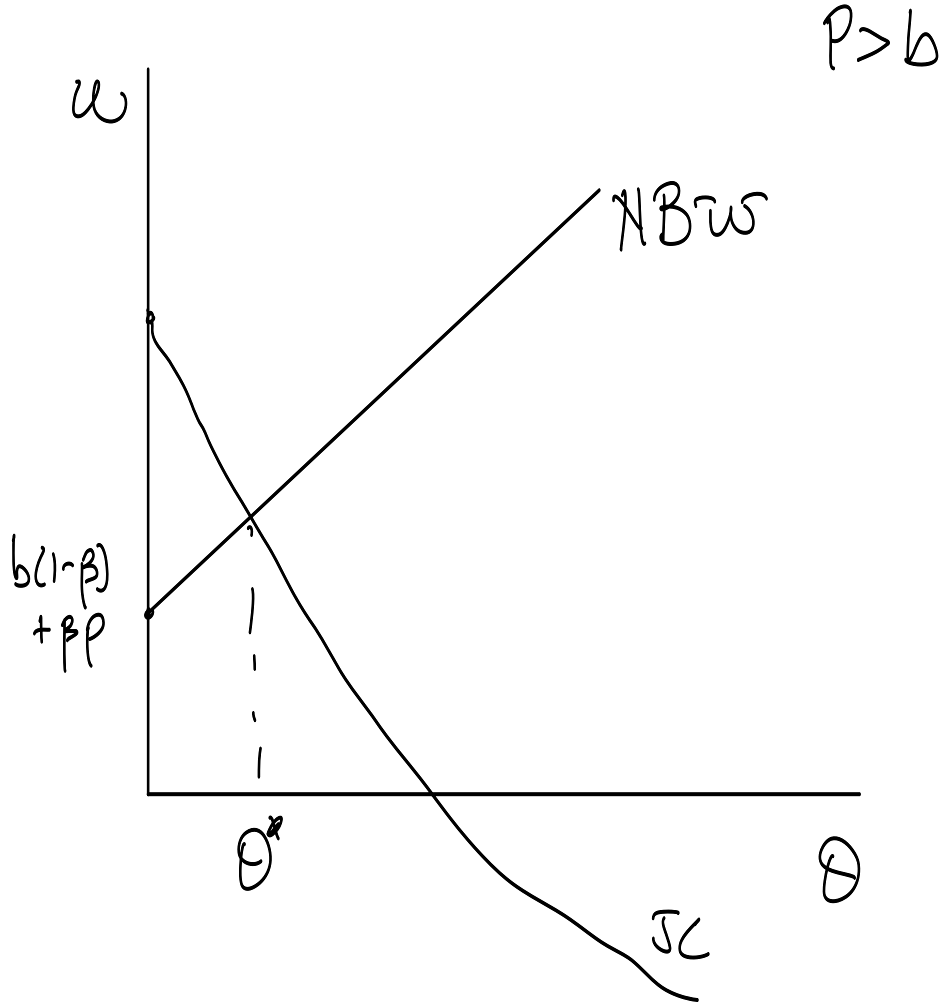

Figure 4: Equilibrium in the w-θ space

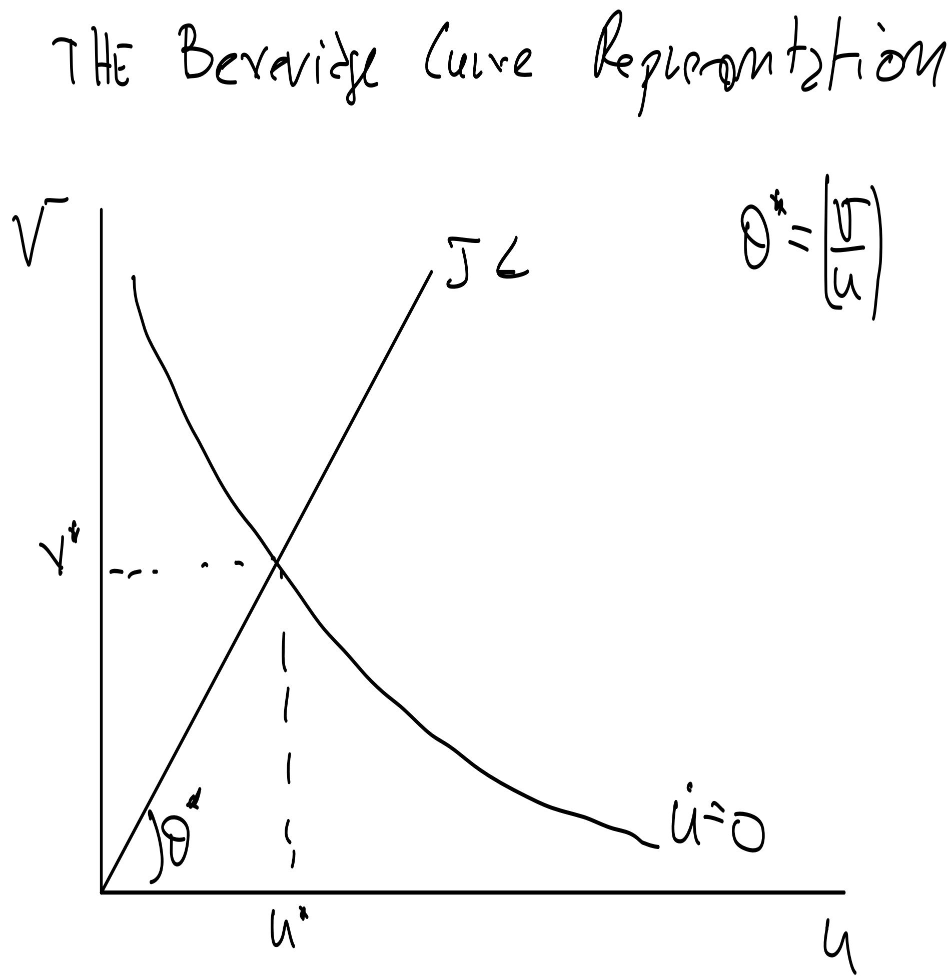

Figure 5: The Beveridge Curve Representation of the Equilibrium

Existence

$$\underbrace{(p-b)(1-\beta) - \beta c\,\theta}_{LHS} = \frac{(r+\lambda)c}{\underbrace{q(\theta)}_{RHS}}$$

$$\frac{\partial RHS}{\partial \theta} = -\frac{(r+s)\,\overbrace{q'(\theta)}^{<0}}{q(\theta)^2} > 0$$

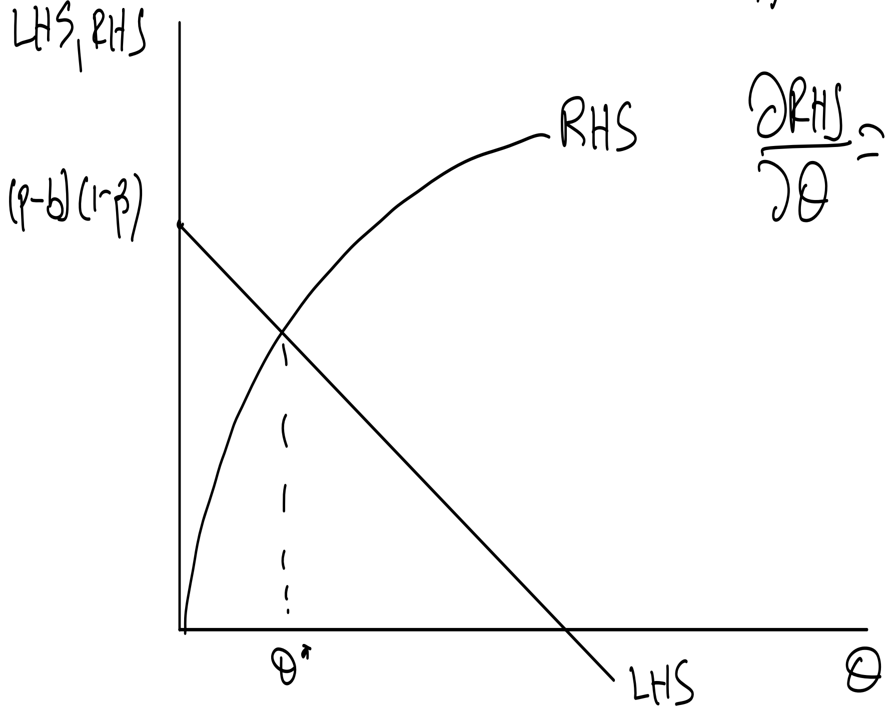

Figure 6: Existence of the Basic Search Equilibrium

COMPARATIVE STATIC:

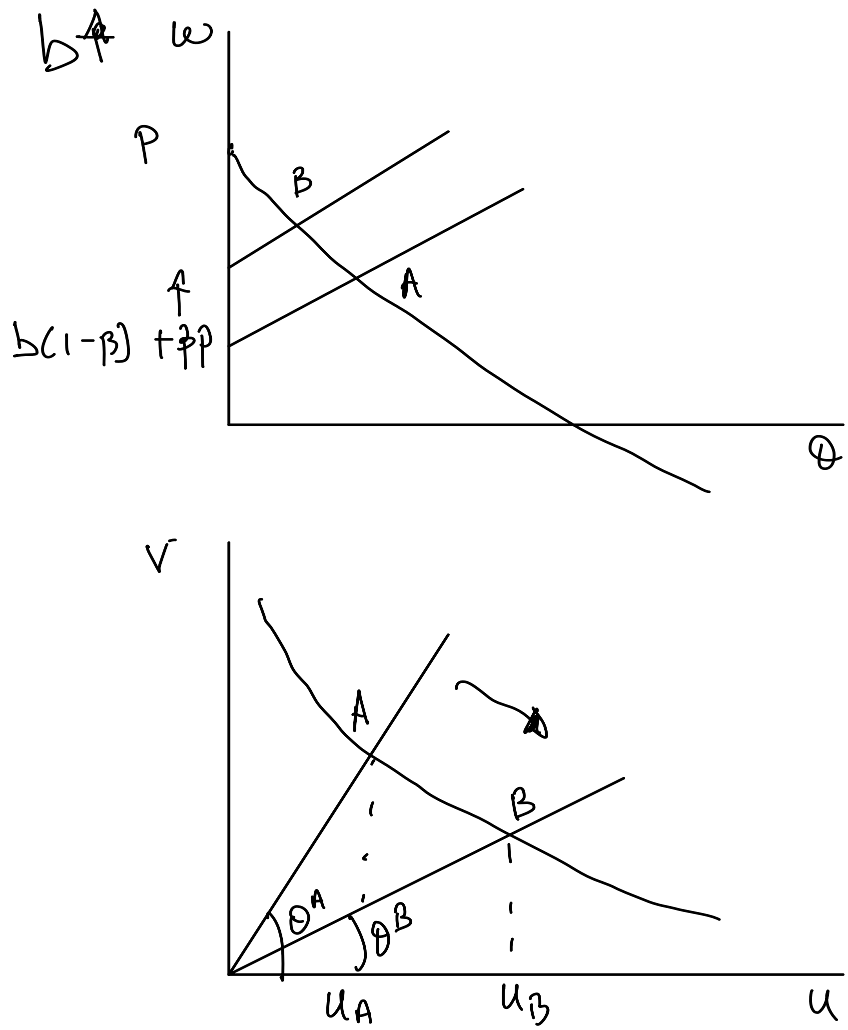

Figure 7: Existence of the Basic Search Equilibrium

3.5 Search Equilibrium as a Fixed Point

- The baseline search equilibrium that is traditionally described as a solution for θ in a block recusive system in which unemployment follows a alst variable.
- The reduced form is obtained by the three equations

$$
\begin{aligned}
w &= p - \frac{(\lambda + r)pc}{q(\theta)} \\
w &= b(1-\beta) + \beta p(1 + c\theta) \\
u &= \frac{\lambda}{\lambda + \theta q(\theta)}
\end{aligned}
$$

- Some notable search scholars argue that the search equililbrium is rather described by a fixed point problem of the type

$$rU = \Gamma(rU) \tag{33}$$

  - the logic of the argument is mathematically very rigorous and also fascinating and elegant.
  - It not difficult to derive an exact expression for the function Γ and the fixed point problem.
  - The fundamental three equations for determining the function Γ in the fixed point problem are

$$
\begin{aligned}
rU &= b + \theta q(\theta)\beta S \\
S &= \frac{p - rU}{r + \lambda} \\
\frac{c}{q(\theta)} &= (1-\beta)S
\end{aligned}
$$

  and finally unemployment is determined by using the fixed point rU^*

$$u = \frac{\lambda}{\lambda + \theta(rU^*)q(\theta(rU^*))}$$

- THe logic of this argument is left to Problem set 2.
- Note that the beauty of this result is that the function Γ is defined in the set

$$rU \in (b, p]$$

  that has an obvious deep economic interpretation

4 Alternative Wage Modeling: Credible Bargaining

- A Model of Strategic Bargaining
- Hall and Milgrom 2008
- We just provide some intuition and few equations
- THe source is the Wasmer book (Appendix chapter 1)
- We need two concepts
  1. **threat point** is the position of disagreemnt
  2. **outside option** is what the party get when negotiation break down
- In NBW (Nash Bargaining Wage)
  $$\text{threat point} = \text{outside option}$$
- in CB (Credible Bargaining) two key assumptions
  1. $$\text{threat point} \neq \text{outside option}$$
  2. Bargaining takes time
- In addition, parties have possibilities of makin offers and counter offers without fully abandoning the barganing

- **Acceptable Offer**. it is an offer that leave the other side indifferent between accepting the offer or making a counter offer.
- An **Equilibrium wage** is a wage that implies an acceptable offer for both parties
- **Key Result** The value will be less dependent on the value of unemployment
- Just some notation
  - z is the worker income during negotiation
  - ξ is the firm cost of delaying production
  - ψ is the probability that bargaining breaks down for exogenous reasons.
  - M is the number of sub-periods of negotiation
  - $r' = \frac{r}{M}$ is the interest/discount rate in sub periods
- Value functions of acceptable offers
- the value to a worker of accepting a firm offer and working at time t
  -
    $$W^w_{n,t}$$
  - THe firm try to make the worker indifferent between accepting and rejecting, and we thus need to calculate the value of rejecting
  - THe key principle in the derivation of equilibrium is

    Payoff of Accepting an offer w = payoff of rejecting w and tomorrow offering to the firm w'

    $$W^w_{n,t} = \psi W_{u,t} + (1-\psi)\left[z + \frac{1}{1+r'} E_t W^{w'}_{n,\frac{t+1}{M}}\right] \tag{34}$$

    where $W^{w'}_{n,\frac{t+1}{M}}$ is the value of employment when rejecting a firm offer and making a counter offer w'

- Let's go to firm. What is an acceptable offer done by the worker
- The value to the firm os accepting a worker offer $'$ is

$$J^{w'}_{\pi,t}$$

- Tee value of rejecting the worker offer and making a acounter offer

$$J'_{\pi,t} = \psi J_{v,t} + (1-\psi)\left[-xi + E_t\left[\frac{1}{1+r'} J^{w}_{\pi,\frac{t+1}{M}}\right]\right]$$

- On of the nice feature of the model is that if

$$\psi \leftarrow 1; \text{then} \qquad \text{CB=NBW}$$

- For the game theory oriented student, the details of the equilibrium are
  - Wasmer-Petrosy Nadeu, Chapter 1 page 31
  - Wasmer-Petrosy Nadeu, Appendix Chapter 1 for the steady state
- The original paper is Hall-Milgrom (2008) American Economic Review, *Thelimited influence of unemployment on wage wage bargain*

5 Basic Model: Discrete Time Version

5.1 Stationary Values

- Sometime the model is presented in discrete time and it is important to manage both reasonably well
- In discrete time the discount rate is

$$\beta = \frac{1}{1+r} < 1$$

- THe bargagining share is ϕ
- In the discrete time stationary
  - The value of the job is time invariant

$$J_{t+1} = J_t = J$$

  - The search value of unemployment is time invriant

$$U_{t+1} = U_t = U$$

  - Market tightness is time invariant

$$\theta_{t+1} = \theta_t = \theta$$

- Let's write the firm asset value functions

$$J = p - w + \beta[sV + (1-s)V] \tag{35}$$

$$V = -c + \beta\left[q(\theta)J + (1 - q(\theta)V\right] \tag{36}$$

- Obviously

$$V = 0 \qquad J = \frac{c}{\beta q(\theta)} \tag{37}$$

- How about the worker

$$U = b + \beta\left[\theta q(\theta)W + (1 - q(\theta))U\right] \tag{38}$$

$$W = w + \beta\left[sU + (1 - s)U\right] \tag{39}$$

- From the firm equation we can write

$$J = \frac{p - w + \overbrace{\beta s V}^{0}}{1 - \beta(1 - s)} \tag{40}$$

  - Just recall the continuous time counterpart as

$$J = \frac{p - w}{r + s}$$

- and the worker

$$W = \frac{w + \beta s U}{1 - \beta(1 - s)} \tag{41}$$

5.2 A Hint on Question 2, Problem Set 2

- you need to prove

$$w = b(1-\phi) + \phi(J + W - U$$

where NBW implies

$$W_u = \phi\left[J + W - U\right]$$

- A bit of a hint from the unemployment value U.
  - Start from the unemployment value

$$U = b + \beta\left[\theta q(\theta)U + (1-\theta q(\theta))W\right]$$

Subtract from both side βU and write it as

$$U(1-\beta) = b + \beta\left[\theta q(\theta)W + (1-\theta q(\theta)U)\right] - \beta U \tag{42}$$

or as

$$U(1-\beta) = b + \beta\left[\theta q(\theta)(W-U)\right] \tag{43}$$

and then use Nash bargaining ......

5.3 Discrete Time Non Stationary

- The discount rate is the same $\beta < 1$
- ϕ is the bargaining share
- In the case of non stationary

$$J_t \neq J_{t+1}; \quad V_t \neq V_{t+1}$$

- The value function for J_t reads

$$J_t = p_t - w_t + \beta \left[\overbrace{sE_t V_{t+1}}^{0} + (1-s) E_t J_{t+1} \right] \tag{44}$$

- Free entry implies that the value of vacancy is zero at all times

$$V_t = 0 = V_{t+1} = V_{t+1} = 0 \qquad \forall t$$

  - where typically one can indicated

$$E_t J_{t+1} = J_{t+1}^e$$

- The value of the vacancy at time t is

$$\overbrace{V_t}^{0} = -c + \beta \left[q(\theta_t) J^e_{t+1} + (1 - q(\theta_t)) \overbrace{V^e_{t+1}}^{0} \right] \tag{45}$$

  - Free entry implies

$$\frac{c}{q(\theta_t)} = \beta J^e_{t+1}; \quad \forall t \tag{46}$$

  - and clearly it is also true that

$$\frac{c}{q(\theta_{t+1})} = \beta J^e_{t+2} \tag{47}$$

  - Now we can write forward the value of 44 for J^e_{t+1} so that

$$J^e_{t+1} = p_{t+1} - w_{t+1} + \beta(1-s) J^e_{t+2} + \beta(1-s) \overbrace{\frac{c}{\beta q(\theta_{t+1})}}^{\frac{c}{\beta q(\theta_{t+1})} = J^e_{t+2}}$$

- We arrive at the fundamental job creation condition in discrete time

$$\frac{c}{q(\theta_t)} = \beta \left[p_{t+1} - w_{t+1} + \frac{(1-s)c}{q(\theta_{t+1})} \right] \tag{48}$$

- This is the **Job Creation in Discrete Time NOn stationary**
  - Note that in the previous equation

$$\theta_{t+1} \neq \theta_t$$

  - For a given set of sequences

$$\{p_{t+1}\}_{t=0}^{\infty}; \quad \{w_{t+1}\}_{t=0}^{\infty};$$

    equation 48 is a fundamental non linear difference equation
  - The equation is stochastic

6 A Couple of "Street Smart" tricks for search scholars and students

6.1 Search Equilibrium in two equations

- There is simple way to think about the model in continuous time.
- They key idea is to consider the model in continuous time with fixed wage so that

$$w = \omega$$

- The model becomes then a pure labor demand model and one does not has to think/bother about the workers' asset equation.
  - It is a bit cheating, since it becomes a 1 sided search model in which only firms play a role.
  - In some sense it is the opposite of the Mc-Call search unemployment model.
- What is the value of a job ?

$$J = \frac{p - \omega}{r + \lambda}$$

- And of course free entry implies

$$V = 0; \quad \Longrightarrow J = \frac{c}{q(\theta)}$$

- The search equilibrium becomes

$$\frac{c}{q(\theta)} = \frac{p - \omega}{r + \lambda} \tag{49}$$

$$u = \frac{\lambda}{\lambda + \theta q(\theta)} \tag{50}$$

where the model makes sense for $b \leq \omega \leq p$

6.2 How to INtegrate SAM Model in a growth model with capital

- Take continuous time model + Acemoglu notes.
- Start from a model with fixed wage (we add NBW later)
- Start from pure labor demand

$$J = \frac{p - w}{r + s}$$

and

$$rV = -c + q(\theta)[J - V]$$

so that

$$V = 0; \quad \Longrightarrow \quad J = \frac{c}{q(\theta)}$$

- How do you add capital ?
  - p is the per worker productiivty
  - Take a neoclassical technology

$$Y = AF(K, N)$$

    with constant returns tu scale
  - Output per capita

$$\frac{Y}{N} = AF(\frac{K}{N}, 1) = Af(k)$$

  - Thus

$$p = y = \frac{Y}{N} = Af(k); \quad \text{This is productivity}$$

- k is capital per worker and we assume
  - k is perfectly reusable (net of depreciation)
  - k is perfectly flexible
- The actual value of a jobs is thus $J + k$ and it follows that

$$r\left[J + k\right] = Af(k) - \delta k - w + s\left[V - J\right] \tag{51}$$

- The firm has technical two assets
  1. k capital, that is also an input in production
  2. the value of being matched to a worker
- How do you chook optimal amount of capital with perfect reusability (simply differentiate 51 with respect to k
- $$\frac{\partial (J + k)}{\partial k} = 0$$

  so that

  $$r = Af'(k) - \delta; \quad \text{uniquely solves for k}$$

  and

  $$k^* = k^*(A, \delta, r)$$

- Incidentally (for those who will take the growth course)

$$Af'(k) = r + \delta \qquad \text{Modified golden rule}$$

- Then we also know that

$$J = \frac{c}{q(\theta)}$$

- At the optimum it must be true that

$$(r + s)\underbrace{J}_{\frac{c}{q(\theta)}} = Af(k^*) - (r + \delta)k^* - w$$

  which leads to

$$(r + s)\frac{c}{q(\theta)} = Af(k^*) - (r + \delta)k^* - w$$

  that gives θ once you have k^*

- If you want to add wages we have

$$Af'(k) = r + \delta; \quad \text{solves for } k^* \tag{52}$$

$$Af(k) - (r+\delta)k - w = \frac{(r+s)c}{q(\theta)} \quad \text{solve for } w \text{ and } \theta \tag{53}$$

$$w = (1-\beta)z + \beta\left[Af(k) - (r+\delta)k + c\theta\right] \quad \text{solve for } w \text{ and } \theta \tag{54}$$

$$u = \frac{s}{s + \theta q(\theta)} \tag{55}$$