UNIVERSITÀ DI TORINO

The Baseline Matching Model: Efficiency

[Sem0057]

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Contents

1 Introduction

- Suppose that in the equilibrium $\theta \uparrow$
- What happens to welfare of the workers?
  - Workers' welfare goes up, since it's easier to find a job.
  - THis is an exmple of a **thick market externality**
- What happens to firms' welfare?
  - welfare (profits) of the firms fall, as it is harder to find a worker
  - This is an example of **congestion externality**
- In the search equilibrium there are trading externalities, since agents utility depends on the relative number of traders in the market. An additional participant on the same side decreases an individual effort (*congestion effect*) while an additional participant of the other side increase welfare (*strategic complements*)
- Does the decentralized market wage internalize these externalities ?
- The natural question to ask in whether the unemployment resulting from the decentralized equilibrium is constrained-efficient, in the sense that it maximizes net output of the aggregate economy.
- We will learn that in general the answer to this question is no. Yet, search scholars discover tha under particular conditions (eventually known in the literature as the Hosios conditions), the decentralized equilibrium can be efficient.

- To answer those questions we need a measure of social welfare
- The key concept is the **Constrained Central Planner**
  - It is a central planner that can move around people with "brute force" (without the makret) but it is constrained in using the matching function.
- We do this in several steps
  1. Social welfare (SW) in a model with exogenous job destrcution and $r = 0$
  2. Social Welfare with exogenous job destruction but a fully forward looking model
  3. Social welfare with endogenous job destruciton (we will get back once we are done with endogenous job destruction)

2 A First Take on the Hosios COnditions: A Micky Mouse Basline Model

- We begin with a measure of social welfare when $r = 0$
- The concept is net output in steady state with $r = 0$

$$SS = \underbrace{(1-u)p}_{\text{Employed Output}} + \underbrace{ub}_{\text{unemployment output}} - \underbrace{cv}_{\text{Cost of vacancies}} \tag{1}$$

- We can always say that

$$cv \equiv c\theta u; \qquad \text{since} \quad \theta = \frac{v}{u}$$

- Further we have a constraint that

$$u = \frac{s}{s + \theta q(\theta)} \tag{2}$$

- Basically we can set up the problem a constrained optimization problem.

- The problem is

$$Max_{\theta,u} SS = (1-u)p + ub - c\theta u \tag{3}$$

$$\text{s.t.} U = \frac{s}{s+\theta q(\theta)} \tag{4}$$

- The Lagrangean is then

$$Max_{\theta,u,\mu} \mathscr{L} = (1-u)p + ub - c\theta u + \mu \left[u - \frac{s}{s+\theta q(\theta)} \right]$$

- The first order conditions are

$$\frac{\partial \mathscr{L}}{\partial u} = 0; \quad -p + b - c\theta + \mu = 0 \tag{5}$$

$$\frac{\partial \mathscr{L}}{\partial \theta} = 0; \quad -cu + \mu \frac{q(\theta) + \theta q'(\theta)}{(s+\theta q(\theta))^2} s = 0 \tag{6}$$

$$\frac{\partial \mathscr{L}}{\partial \mu} = 0; \quad u = \frac{s}{s+\theta q(\theta)} \tag{7}$$

- Note that in the second equation unemployment can be simplified from both terms so that one has

$$cu = \mu(q(\theta + \theta q'(\theta)) \underbrace{\frac{s}{s+\theta q(\theta)}}_{u} \frac{1}{s+\theta q(\theta)}$$

or

$$\mu = \frac{c(s+\theta q(\theta)}{(q(\theta) + \theta q'(\theta))}$$

- From the first equation

$$\mu = (p-b) + c\theta$$

- Substituting out μ one has

$$c(s+\theta q(\theta)) = (p-b)\left(q(\theta)+\theta q'(\theta)\right) + \theta c\left[q(\theta)+\theta q'(\theta)\right]$$

- Let's recall some definition we had on $q(\theta)$

$$\underbrace{\eta(\theta)}_{\text{Absolute Value fo Elasticity of the matching function}} = \left| -\frac{\frac{dq}{q}}{\frac{d\theta}{\theta}} \right| = -\frac{q'(\theta)}{q(\theta)}\theta$$

  - general with CRS matching function

$$0 \leq \eta(\theta) \leq 1$$

  - Further if $q = \theta^{-\alpha}$
  -

$$\eta(\theta) = \alpha; \quad \text{independent of } \theta$$

- Recall also that

$$\frac{\partial \theta q(\theta)}{\partial \theta} = q(\theta) + q'(\theta)\theta = q(\theta)\left[1 + \frac{\theta q'(\theta)}{q(\theta)}\right]$$

  or

$$\frac{\partial \theta q(\theta)}{\partial \theta} = q(\theta)(1-\eta(\theta))$$

- it follows

$$c(s+\theta q(\theta)) = (p-b)\underbrace{(q(\theta)+\theta q'(\theta))}_{q(\theta(1-\eta(\theta))} + \theta c \underbrace{[q(\theta)+\theta q'(\theta)]}_{q(\theta(1-\eta(\theta))}$$

- And dividing by $q(\theta)$

$$c\frac{(s+\theta q(\theta))}{q(\theta)} = (p-b)(1-\eta(\theta)) + \theta c(1-\eta(\theta))$$

- We thus have a single equation in (θ)

$$(p-b)(1-\eta(\theta)) + \theta c - \eta(\theta)\theta c = \frac{cs}{q(\theta)} + c\theta$$

or

$$(p-b)(1-\eta(\theta)) - \eta(\theta)\theta c = \frac{cs}{q(\theta)}; \quad \text{Central Planner Solution} \tag{8}$$

- We call θ^*_{CP} the efficient solution for θ

- Let's go back to the decentralized SAM model with NBW in continuous time
- At some point with showed that the general equilibrium i obtained by a single equation in θ

$$(p-b)(1-\beta)-\beta c\theta=\frac{(r+s)c}{q(\theta)}$$

but we need to have with $r=0$ for a full comparison

- THe market solution is thus

$$(p-b)(1-\beta)-\beta c\theta=\frac{(r+s)c}{q(\theta}\quad \text{Decentralized Solution} \tag{9}$$

- We call θ^*_{MK} the decentralized solution for θ
- Let's compare the two (with $r=0$)

$$(p-b)(1-\beta)-\beta c\theta=\frac{sc}{q(\theta)}\quad \text{Decentralzied Solution} \tag{10}$$

$$(p-b)(1-\eta(\theta))-\eta(\theta)\theta c=\frac{cs}{q(\theta)};\quad \text{Central Planner Solution} \tag{11}$$

- Under what conditions do we have that Market Solution= Central Planner Solution?

$$\underbrace{\beta}_{\text{Bargaining Share of the Worker}} = \underbrace{\eta(\theta)}_{\text{Elasticity of the Matching Function}} \quad \text{Hosios Conditions} \tag{12}$$

- What are the two elements of the Hosios Condition?

  1. $0 \leq \beta \leq 1$;   Strucutral Parameter of Negotiation
  2. $0 \leq \eta(\theta) \leq 1$;   Strucutral Parameter of matching Function

- If the matching function is Cobb Douglas

$$\beta = \alpha \quad \text{Hosios Conditions with Cobb Dougals Matching Function}$$

- There is no ecomomic mechanism that imply that the these 2 parameters should be identical
- The idea is that at the table of **negotiation, the parties just look at their own surplus and do not internalize the effect of their bragaining on the wlefare of the agents who are outside (unemployed and vacancies)**
- Only under the (non obvious) Hosios conditions such condition is satisfied
- Note also that here we are using the Hosios NBW. We said that there are many wages that satisfy the search equilibrium. Among allo those wages, the net output is clearly inefficient.

3 Efficiency In The Baseline Forward Looking Model (Exogenous Destruction)

- We now generalize the simple exercise with a proper forward looking problem.
  - We are still with exogenous job destrution but now $r > 0$
- The measure of social output is the present discounted value of

$$\Omega = \int_0^\infty e^{-rt}[p(1-u) + zu - pc\theta u]dt$$

  where output is the sum of three components: total production, total unemployment income minus average search cost.

- The social planner maximizes Ω subject to the matching frictions imposed by the matching technology. Basically, the problem is

$$\begin{aligned} Max_{u,\theta} & \quad \int_0^\infty e^{-rt}[p(1-u) + zu - pc\theta u]dt \\ \text{s.t. } u\theta q(\theta) &= \lambda(1-u) \end{aligned}$$

- Dynamic optimization is illustrated in detail in the growth course.
- To solve this problem form the current value Hamiltonian as

$$H = e^{-rt}[p(1-u) + zu - pc\theta u] + \phi[-u\theta q(\theta) + \lambda(1-u)]$$

  - whose first order conditions are

$$\begin{aligned} \frac{\partial H}{\partial \theta} &= 0. \\ \frac{\partial H}{\partial u} &= -\dot{\phi} \end{aligned}$$

    (note that θ is a control variable and u is a state variable).
  - The first order conditions in our model are respectively

$$-e^{-rt}pcu - \phi(1-\eta(\theta))q(\theta)u = 0$$

$$-e^{-rt}[p - z + pc\theta] - \phi(\lambda + \theta q(\theta)) = -\dot{\phi}$$

- From the first equation we obtain

$$\phi = -\frac{pce^{-rt}}{q(\theta)[1-\eta(\theta)]}; \qquad \dot{\phi} = \frac{pce^{-rt}}{q(\theta)[1-\eta(\theta)]}$$

  where $\eta(\theta)$ is the elasticity of the matching function with respect to θ.

- This $\dot{\phi}$ can be substituted into the first equation to obtain

$$-[p-z+pc\theta]e^{-rt} + \underbrace{\frac{pce^{-rt}}{q(\theta)[1-\eta(\theta)]}(\lambda+\theta q(\theta))}_{\phi} = -\underbrace{\frac{rpce^{-rt}}{q(\theta)[1-\eta(\theta)]}}_{\dot{\phi}}$$

  which simplifies to

$$-q(\theta)(1-\eta(\theta)(p-z+c\theta) + pc(\lambda+\theta q(\theta)) = -rpc \tag{13}$$

  and

$$(p-z)[1-\eta(\theta)] - pc\theta\eta(\theta) = \frac{(\lambda+r)}{q(\theta)}pc \tag{14}$$

- How does equation 14 compare with the social optimum?
  - To see this recall the firm's problem in the decentralized search equilibrium

$$\frac{(r+\lambda)pc}{q(\theta)} = p - w$$

  - where the wage rate is obtained via decentralized bargaining or as the solution to

$$w = (1-\beta)z + \beta[p + pc\theta]$$

    so that the supply of jobs in a decentralized equilibrium is

$$(p-z)(1-\beta) - pc\theta\beta = \frac{(r+\lambda)pc}{q(\theta)} \tag{15}$$

    Comparing the social outcome with decentralized outcome, it is clear that the decentralized bargaining maximize social net output if

$$\beta = \eta(\theta)$$

    or if the sharing parameter is identical to the elasticity of the matching function.

4 Policy Instruments In The Baseline Model (Exogenous Destruction)

Wage Taxes

* Let w be the wage paid when there are no taxes, so that labor costs are equal to the net wage.
* τ is a wage subsidy so that the wage would become $w + \tau$
* there is a marginal income tax equal to t so that the net wage received by the worker is $(1 - t)(w + \tau)$
* the total tax paid is then $T(w) = w - (1 - t)(w + \tau) = tw - \tau(1 - t)$

Wage taxes are linear and a smooth function of income. If the gross wage is w the net wage received by a worker is $(1 - t)(w + \tau)$.One can think of a tax subsidy τ received by the worker and subsequently being taxed at t. The tax paid by the worker is tw minus the net subsidy received $\tau(1 - t)$. Total tax is

$$T(w) = tw - \tau(1 - t)$$

While t is the marginal tax rate, the tax subsidy makes τ makes the tax system progressive ($\tau > 0$), purely proportional ($\tau = 0$) or regressive ($\tau < 0$). To see this not that

$$\frac{T(w)}{w} = t - \frac{\tau(1 - t)}{w}$$

Employment Subsidy

employment is subsidized at rate a per job. The labor product become $p + a$

Hiring Subsidy

Firms receive a hiring subsidy once and for all when the worker is hired The value of the subsidy is pH

Firing Taxes

Firms pay a firing tax when a separation takes place. The value of the firing tax is pT Note that the firing tax is not a transfer to the worker (a pure severance payments has no effect in equilibrium)

Unemployment Compensation

The policy parameter is the after tax replacement rate, or the ratio of net unemployment benefit to the average net income from work. b is defined as

$$\begin{aligned} b &= \rho[w - T(w)] \\ b &= \rho[(1 - t)w + \tau(1 - t)] \end{aligned}$$

Obviously b is not taxed since it is already defined in net terms.

VALUE FUNCTIONS WITH POLICY

The value of unemployment is given by

$$rU = z + \rho[(1 - t)w + \tau t] + \theta q(\theta)[W - U]$$

The employed worker net worth in a job that pays w_j is

$$rW_j = (1 - t)w_j + \tau(1 - t) + \lambda[U - W_j]$$

The value of a vacancy is

$$rV = -pc + q(\theta)[J_j + pH - V]$$

where the firms, conditional upon job formation, gains a value of a job plus the one time hiring subsidy. The value of a filled job is

$$rJ_j = p + a - w_j + \lambda[-pF - J]$$

where $-pF$ is the firing tax to be paid in case of job separation. The free entry condition implies

$$J_j + pH = \frac{pc}{q(\theta)}$$

4.1 WAGE DETERMINATION

Wages are chosen to maximize the Nash product and can be renegotiated at any time. When there are hiring subsidies and firing taxes the application of the wage rule is more problematic.

There are two important issues to discuss: the two-tier regime and the role of taxation.

On impact, when they first meet, the initial wage is determined by

$$B_o = (W_j - U)^{\beta}(J_j + pH - V)^{1-\beta}$$

where $J + pH$ is the gain to the firm from wage agreement.

When the employment relationship is undergoing, hiring subsidy are no longer available but the firm has to pay the firing tax if the negotiation breaks does. This implies that the wage rule is now

$$B_1 = (W_j - U)^{\beta}[J_j - (V - pF)]^{1-\beta}$$

where $V - pF$ is the firm's threat point in the negotiation rule. There will be a two-tier wage regime where, w_o is the "outside" wage that solves the Nash product B_o while w_1 is the inside wage that solves B_1 Formally

$$\begin{aligned} w_o &= \arg\max B_o \\ w_i &= \arg\max B_1 \end{aligned}$$

The solution to w_o solves (by taking logs of B_o and taking the derivative with respect to w_o)

$$\frac{\beta}{W_j - U}[\frac{\partial W_j}{\partial w_{oj}} - \frac{\partial U_j}{\partial w_{oj}}] + \frac{1-\beta}{J_j + pH - V}[\frac{\partial J_j}{\partial w_{oj}} - \frac{\partial V_j}{\partial w_{oj}}] = 0$$

Similarly the solution to w_{ij} is

$$\frac{\beta}{W_j - U}[\frac{\partial W_j}{\partial w_{ij}} - \frac{\partial U_j}{\partial w_{ij}}] + \frac{1-\beta}{J_j + pF - V}[\frac{\partial J_j}{\partial w_{ij}} - \frac{\partial V_j}{\partial w_{ij}}] = 0$$

Since there are taxes, the derivative of the value functions do not cancel out. While $\frac{\partial U_j}{\partial w_{ij}} = \frac{\partial U_j}{\partial w_{oj}} = \frac{\partial V_j}{\partial w_{oj}} = \frac{\partial V_j}{\partial w_{ij}} = 0$ with respect to the specific wage j (note that is only the average wage that enters the value functions).

$$\begin{aligned} \frac{\partial W_j}{\partial w_{ij}} &= \frac{\partial W_j}{\partial w_{oj}} = \frac{1-t}{r+\lambda} \\ \frac{\partial J_j}{\partial w_{ij}} &= \frac{\partial J_j}{\partial w_{oj}} = \frac{1}{r+\lambda} \end{aligned}$$

So that the sharing rules become

$$\begin{aligned} (1-\beta)(W_j - U) &= \beta(1-t)(J_j + pH - V) \qquad \text{for } w_{oj} \\ (1-\beta)(W_j - U) &= \beta(1-t)(J_j + pF - V) \qquad \text{for } w_{ij} \end{aligned}$$

This implies that the share of the surplus going to the worker depends on t. To see this (add and subtract $\beta t[W - U]$) to the outside surplus(it's the same for the inside wage) to obtain

$$W_j - U = \beta(1-t)(J_j + pH - V) + \beta(W_j - U) + \beta t(W_j - U) - \beta t(W_j - U)$$

to obtain

$$\begin{aligned} W_j - U &= \frac{\beta(1-t)}{1-\beta t}[J_j + pH - V + W_j - U] \\ W_j - U &= \frac{\beta(1-t)}{1-\beta t} B_o \end{aligned}$$

so that the worker's share of the surplus is now $\frac{\beta(1-t)}{1-\beta t}$ which is a decreasing function of t. The marginal tax rate influences the division of the surplus from a job, whereas average tax rates and subsidies influence the outcome of the bargain only through their effect on the surplus shared. An increase in t reduces the worker's share of the surplus, but to see whether wages actually fall depend on what happens to the total surplus and to the alternative return to labor. The idea is that the marginal tax rate system imposes a joint loss to the firm that can be reduced by keeping wages low.

4.1.1 OUTSIDE AND INSIDE WAGE

To obtain the outside wage proceed as follows

$$(r+\lambda)[W_j - U] = (1-t)w_j - \rho[(1-t)w + \tau t] - \theta q(\theta)[W-U]$$

Imposing symmetry $w = w_j$ (all outside and inside wages are the same in equilibrium) and using the free entry condition

$$W - U = \frac{\beta(1-t)}{1-\beta}\frac{cp}{q(\theta)}$$

one obtains

$$(r+\lambda)[W-U] = (1-t)w + \tau(1-t) - z - \rho[(1-t)w + \tau t] - \frac{\theta\beta(1-t)cp}{1-\beta}$$

while

$$(r+\lambda)(J+pH) = p + a - w - \lambda pF + (r+\lambda)pH$$

So that the satisfies

$$(1-\beta)[(1-t)w + \tau(1-t) - z - \rho[(1-t)w + \tau t] - \frac{\theta\beta(1-t)cp}{1-\beta}] =$$
$$\beta(1-t)[p + a - w - \lambda pF + (r+\lambda)pH]$$

Dividing everything by $(1-t)$ and collecting terms

$$w[1-\rho(1-\beta)] = \frac{z(1-\beta)}{1-t} - (1-\beta)\tau + (1-\beta)\rho + \beta[p + a - \lambda pF + (r+\lambda)pH + \theta cp]$$

which simplifies to the final expression as

$$w_o = \frac{1-\beta}{1-\rho(1-\beta)}[\frac{z}{1-t} - \tau(1-\rho)] + \frac{\beta}{1-\rho(1-\beta)}[p(1+\theta c + (r+\lambda)pH - \lambda F) + a] \quad \text{(outside wage)}$$

To get the inside wage proceed in a similar way using

$$(r+\lambda)(J+pF) = p + a - w - \lambda pF + (r+\lambda)pF$$

and substituting into the sharing rule one obtains

$$w_i = \frac{1-\beta}{1-\rho(1-\beta)}[\frac{z}{1-t} - \tau(1-\rho)] + \frac{\beta}{1-\rho(1-\beta)}[p(1+\theta c + rF) + a] \quad \text{(inside wage)}$$

Differences among wages

- ∗ if there are no hiring subsidies ($H = 0$) the wage increases after renegotiation
  - ∗ the inside wage does not depend on the hiring subsidy (which is already been paid)
  - ∗ if $H = F$ the two wages are the same
  - ∗ $w_o < w_i$ if $F > H$

Similarities

- ∗ the replacement rate increases wages since it increase the outside option
  - ∗ the employment subsidy increase wages
  - ∗ the net tax subsidy receive by the match is $\tau(1-\rho)$ so that they increase the wage, the employed worker gets the full subsidy while the unemployed gets only a fraction ρ
  - ∗ The marginal tax rate influences the wage to the extant that specific unemployed income z is not taxed
  - ∗ If $z = 0$ and $\tau = 0$ (proportional taxation) the tax system does not introduce any distortion
  - ∗ If $z = 0$ and $\tau > 0$ (progressive taxation) the tax system reduces the negotiated wage

4.2 EQUILIBRIUM WITH POLICY

Labor demand is obtained from

$$\frac{pc}{q(\theta)} - pH = \frac{p + a - w - \lambda pF}{r + \lambda}$$

so that the expression is

$$w = p + a + p[(r+\lambda)H - \lambda F] - \frac{(r+\lambda)cp}{q(\theta)} \tag{16}$$

and is a negative relationship between w and θ. The second relationship for obtain θ is the outside wage, since job formation is obtained on the basis of the expected profit from a new job. The outside wage and labor demand (equation 16) solve a unique θ while equilibrium unemployment is simply given by

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

The Search equilibrium with exogenous job destruction and a set of policy instruments (ρ, τ, t, F, H) is a couple (u, θ) and two wage rules $w_o w_i$ satisfying

Free entry ($V = 0$)

Outside Wage bargaining $(1-\beta)[W - U] = \beta(1-t)[J + pH]$

Inside Wage Bargaining $(1-\beta)[W - U] = \beta(1-t)[J + pF]$

Balance Flow $m(u, v) = \lambda(1-u)$

The effects of the various policies. To obtain the effects of the single policies it suffices to substitute the outside into equation 16 the outside wage to obtain

$$p + a + p[(r+\lambda)H - \lambda F] - \frac{(r+\lambda)cp}{q(\theta)} =$$
$$\frac{1-\beta}{1-\rho(1-\beta)}[\frac{z}{1-t} - \tau(1-\rho)] + \frac{\beta}{1-\rho(1-\beta)}[p(1+\theta c + (r+\lambda)pH - \lambda F) + a]$$

Employment subsidies increase market tightness. To see this simply totally differentiate the previous equation with respect to a to obtain

$$1 + \frac{(r+\lambda)cq'(\theta)}{q^2(\theta)}\frac{\partial\theta}{\partial a} = \frac{\beta}{1-\rho(1-\beta)}$$

which simplifies to

$$\frac{(1-\rho)(1-\beta)}{1-\rho(1-\beta)} = -\frac{(r+\lambda)cq'(\theta)}{q^2(\theta)}\frac{\partial\theta}{\partial a}$$

so that

$$\frac{\partial\theta}{\partial a} > 0 \rightarrow \frac{\partial u}{\partial a} < 0$$

Hiring Subsidies increase market tightness

Proceeding as above one obtains

$$\frac{(1-\rho)(1-\beta)}{1-\rho(1-\beta)} = -\frac{(r+\lambda)cq'(\theta)}{q^2(\theta)}\frac{\partial\theta}{\partial H}$$

so that

$$\frac{\partial\theta}{\partial H} > 0 \rightarrow \frac{\partial u}{\partial H} < 0$$

Marginal tax rates reduce market tightness

$$-\frac{(1-\beta)z}{1-\rho(1-\beta)}\frac{1}{(1-t)^2} = \frac{(r+\lambda)cq'(\theta)}{q^2(\theta)}\frac{\partial\theta}{\partial t}$$

so that

$$\frac{\partial\theta}{\partial t} < 0 \rightarrow \frac{\partial u}{\partial t} > 0$$

Note that in a model without home production $z = 0$ taxation is completely neutral.

Tax subsidies increase market tightness

$$\frac{(1-\rho)(1-\beta)}{1-\rho(1-\beta)} = -\frac{(r+\lambda)cq'(\theta)}{q^2(\theta)}\frac{\partial\theta}{\partial\tau}$$

so that

$$\frac{\partial\theta}{\partial\tau} > 0 \rightarrow \frac{\partial u}{\partial\tau} < 0$$

Firing Taxes reduce market tightness

$$\frac{(1-\rho)(1-\beta)}{1-\rho(1-\beta)} = \frac{(r+\lambda)cq'(\theta)}{q^2(\theta)}\frac{\partial\theta}{\partial F}$$

so that

$$\frac{\partial\theta}{\partial F} < 0 \rightarrow \frac{\partial u}{\partial F} > 0$$

Finally, replacement Rate reduce market tightness

$$\frac{\partial\theta}{\partial\rho} < 0 \rightarrow \frac{\partial u}{\partial\rho} > 00$$